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Message of MOMA Journal Editor-In-Chief

This journal concerns both the national and international scientific community and will be primarily focusing on Models and Optimisation of Systems. Systems will be utilized in different applications for example, Web technologies, Information Systems, Decision Systems, Embedded Systems, Control-command Systems and Real-time Systems. Space of journal is also dedicated to mathematical analysis like functional spaces, polynomial computing etc.

This edition contains several parts: parts are dedicated to fundamental mathematics and computer sciences works, another part present mathematical history of SonjaBrentjes of Max Planck Institute of history sciences of Berlin and Glen Van Brummelen of Quest University - Canada). We included mathematical history space because of the importance of this field like argumented by researcher Bennacer El Bouazzati of university of Rabat.

« Toutes les cultures antiques ont tissé des conceptions cosmologiques, à des degrés divers en explication, sur l'ordre de l'univers et la composition des élémentsdont il est constitué. L'observation des phénomènes de la nature permit aux élites cultivées d'élaborer des idées sur le début de l'univers,son âge et les rapports entre ses parties.au cours des échanges entre les cultures qui n'ont cessé de se développer dans le temps, les conceptions cosmologiques ont voyagé à travers les multiples moyens de contact.

La diversité des idées poussa les penseurs à souligner les particularités des conceptions cosmologiques et à comparer les caractéristiques à des fins multiples. Des emprunts et des sélections ont marqué toutes les étapes du développement de ces conceptions. Avec le temps, ces conceptions se sont enrichies de moyens arithmétiques et géométriques pour combiner les éléments et les organiser en séries et en relation causales ; ce qui donna lieu à des registres et des tables qui relatent le cours des faits du monde.

Le déploiement des mathématiques prend un essor remarquable aux V° et IV° siècles avant J.-C, pour aboutir à des connaissances bien agencées. Dans un riche contexte en controverse philosophique et logique.au cours de ces deux siècles,des constructions conceptuelles sophistiquées ont vu le jour, tendant à présenter des images traçant le cours des faits. Entre le IV° et le III° siècles,l'astronomie,l'optique,la mécanique,l'harmonique et des thèmes apparentés deviennent des sciences qui ne peuvent se passer de l'application des mathématiques pour formuler leurs lois en termes de principes et de preuves. Les conceptions cosmologiques,qui se nourrissent des réalisations de ces quatresciences, ne sont pas aisémentmathématisables, parce qu'elles sont imprégnées des croyances religieuses bien ancrées dans les esprits.

Les cosmologies pythagoricienne,aristotélicienne,stoicienne et atomiste, se sont transmises de génération en génération dans un processus de concurrences qui n'a cessé de développer des arguments de plus en plus forts et d'accumuler des observations de plus en plus proches des faits de la nature. Les malentendus entre l'astronomie et la cosmologie se multiplient mais les tentatives de réconciliations deviennent de plus en plus pressantes ;ceci fut souligné par Cléomède et Géminos avant et au temps de Ptolémée.

Martianus et Erigène propose des cosmologies issues de la solutions de Héraclide du Pont pour ce qui concerne la place de Mercure et de Venus dans l'ordre des corps célestes. IbnSina affirme que l'astronomie a besoin d'une cosmologie convenable et Ibn al-Haytham appelle à réformer les défauts de la théorie astronomique en place et à présenter une configuration consistante de l'univers. Des tentatives se multiplient durant des siècles et la synthèse la lus convaincante s'est élaborée au XVII° siècle.

L'histoire de la cosmologie n'a pas connu que les conceptions qui se sont nourries des innovations des sciences voisines, d'autres conceptions furent présentées à titre d'alternatives à celles considérées comme entachées de philosophie, comme la doctrine anwa' et ce qui a été appelé la configurationsunnite du monde. Ainsi, toutes les conceptions ne sont pas du même degré en leur pouvoir de persuasion ; quelques-unes sont dénuées de contenu plausible. Les critiques ne sont pas équivalentes, quelques-unes sont stériles à certains égards. Les conceptions qui ont contribué à présenter une image rationnelle du monde sont celle qui ont combinés les résultats *des sciences qui réussirent à mathématiser convenablement les résultats de l'observation (et/ou l'expérimentation). »*

We would like to express our gratitude to everyone who has contributed towards the success of this edition.

Sincerely yours,
Dr.Mostefa BELARBI

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Second Order Impulsive Functional Differential Equations with Variable Times and State-Dependent Delay

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***Abstract*—In this paper, we establish sufficient conditions for the existence of solutions for a class of initial value problem for impulsive functional differential equations with variable times involving infinite state-dependent delay.**

***Index Terms*—Differential equation, state-dependent delay, fixed point, infinite delay, impulses, variable times.**

I. Introduction

THIS paper deals with the existence of solutions to the initial value problems (IVP for short) for the second differential equations with variable times and state dependent delay of the form,

$$y''(t) = f(t, y_{\rho(t,y_t)}), \text{ a.e. } t \in J = [0, b], \tag{1}$$

$$t \neq \tau_k(y(t)), \ k = 1, \ldots, m, y(t^+) = I_k(y(t)), \tag{2}$$

$$t = \tau_k(y(t)), \ k = 1, \ldots, m, \tag{3}$$

$$y'(t^+) = \overline{I}_k(y(t)), \ t = \tau_k(y(t)), \ k = 1, \ldots, m, \tag{4}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \tag{5}$$

$$y'(0) = \eta, \tag{6}$$

where $f : J \times \mathcal{B} \to \mathbb{R}$, $\rho : J \times \mathcal{B} \to (-\infty, b]$, $I_k, \overline{I}_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \ldots, m$ are given continuous functions, $\phi \in \mathcal{B}$, $y(t^+) = \lim_{h \to 0^+} y(t + h)$ and $y(t^-) = \lim_{h \to 0^-} y(t + h)$ represent the right and left hand limits of $y(t)$ at t and $\mathcal{B}$ is a phase space to be specified later. For any function y and any $t \in J$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t + \theta)$ for $\theta \in (-\infty, 0]$. We assume that the histories y_t belong to $\mathcal{B}$.

The notion of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato [15] (see also Kappel and Schappacher [20] and Schumacher [30]. For a detailed discussion on this topic we refer the reader to the book by Hino *et al*. [18]. For the case where the impulses are absent, an extensive theory has been developed for the problem (1)-(6). We refer to Hale and Kato [15], Corduneanu and Lakshmikantham [9], Hino *et al*. [18], Lakshmikantham *et al* [25].

Impulsive differential equations have become more important in recent years in some mathematical models of real processes and phenomena studied in control, physics, chemistry, population dynamics, biotechnology and economics. There has been a significant development in impulse theory, in recent years, especially in the area of impulsive differential equations with fixed moments, see the monographs of Benchohra *et al.* [5], Lakshmikantham *et al.* [25] and Samoilenko and Perestyuk [29] and the references therein. The theory of impulsive differential equations with variable times is relatively less developed due to the difficulties created by the state-dependent impulses. Some interesting results have been done by Bajo and Liz [1], Benchohra *et al.* [3], [7] and Benchohra and Ouahab [8], Frigon and O'Regan [10], [11], [12], Graef and Ouahab [13], Kaul *et al.* [21], Kaul and Liu [22], [23], Lakshmikantham *et al.* [26], [27] and Liu and Ballinger [28]. The results of the present paper extend those considered in the above cite literature for constant delay. Our approach here is based on the nonlinear alternative of Leray-Schauder type [14].

II. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\| := \sup\{|y(t)| : t \in J\}.$$

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [15] and follow the terminology used in [19], but we will add some transformations. Thus $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$.

$L^1(J,\mathbb{R})$ denotes the Banach space of measurable functions $y : J \longrightarrow \mathbb{R}$ which are Lesbegue integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|dt.$$

$AC^1(J,\mathbb{R})$ denote the space for all differentiable functions whose first derivative is absolutely continuous.

Definition II.1. *The map $f : J \times \mathcal{B} \to \mathbb{R}$ is said to be Carathéodory if:*

(i) *The function $t \longmapsto f(t,u)$ is measurable for each $u \in \mathcal{B}$*
(ii) *The function $u \longmapsto f(t,u)$ is continuous for a.e. $t \in J$.*

Consider the sets

$$PC = \Big\{y : [0,b] \to \mathbb{R} : y \text{ which there exist } 0 < t_1 < t_2 < ... < t_{m+1} = b \text{ such that } t_k = \tau_k(y(t_k^-)) \text{ and } y(t_k^+), y(t_k^-) \text{ exists with, } y(t_k^-) = y(t_k)\ k = 1,\dots,m,\ y_k \in C(J_k,\mathbb{R})\Big\},$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}]$, $k = 1,\dots,m$,
and

$$B_b = \{y : (-\infty, b] : y|_{(-\infty,0]} \in \mathcal{B} \text{ and } y|_J \in PC\}.$$

Let $\|.\|_b$ the seminorm in B_b defined by

$$\|y\|_b = \|y_0\|_{\mathcal{B}} + \sup\{|y(t)| : 0 \le t \le b\},\ y \in B_b.$$

For the definition of the phase space $\mathcal{B}$ we introduce the following axioms.

(A_1) If $y : (-\infty, b) \to \mathbb{R}, b > 0,\ y_0 \in \mathcal{B}$, the following conditions hold :
(i) $y_t \in \mathcal{B}$,
(ii) There exists a positive constant H such that $|y(t)| \le H\|y_t\|_{\mathcal{B}}$,
(iii) There exist two functions $K(\cdot), M(\cdot) : J \to \mathbb{R}^+$, independent of y, with K continuous and M locally bounded such that :

$$\|y_t\|_{\mathcal{B}} \le K(t)\sup\{\ |y(s)| : 0 \le s \le t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) The space $\mathcal{B}$ is complete.

Denote $K_b = \sup\{K(t) : t \in J\}$ and $M_b = \sup\{M(t) : t \in J\}$.

$\mathcal{B} = \{y : (-\infty, 0] \to \mathbb{R}, y$ is continuous every where except for a finite number of points $\bar{t}$ at which $y(\bar{t}^+), y(\bar{t}^-)$ exist and $y(\bar{t}^-) = y(\bar{t})\}$

Definition II.2. *A function $y \in B_b \bigcap \bigcup_{i=1}^m AC^1((t_i, t_{i+1}),\mathbb{R})$ is to be a solution of (1)-(6) if y satisfies $y''(t) = f(t, y_{\rho(t,y_t)})$ a.e $t \in J = [0,b]$, $t \ne \tau_k(y(t))$, $k = 1,\dots,m$, the conditions $y(t^+) = I_k(y(t))$, $y'(t^+) = \overline{I}_k(y(t))$ $t = \tau_k(y(t))$, $k = 1,\dots,m$, and $y(t) = \phi(t)$, $t \in (-\infty, 0]$, $y'(0) = \eta$.*

We are now in a position to state and prove our result for the problem $(1) - (6)$.

III. Existence of Solutions

In this section we will present an existence result for the problem (1)-(6). First, we introduce the following hypotheses.

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times \mathcal{B}, \rho(s,\varphi) \le 0\}$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that $\|\phi_t\|_{\mathcal{B}} \le L^\phi(t)\|\phi\|_{\mathcal{B}}$ for every $t \in \mathcal{R}(\rho^-)$.
(H_1) The function $f : J \times \mathcal{B} \to \mathbb{R}$ is Carathéodory,
(H_2) There exists $p \in L^1(J,\mathbb{R}_+)$ and $\psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that

$$|f(t,u)| \le p(t)\psi(\|u\|_{\mathcal{B}}) \text{ for each } t \in J \text{ and all } u \in \mathcal{B},$$

with

$$K_b\int_0^b p(s)ds < \int_C^\infty \frac{dx}{\psi(x)},$$

where $C = M_b\|\phi\|_B + K_b|\phi(0)|$.

(H_3) The functions $\tau_k \in C^1(\mathbb{R},\mathbb{R})$ for $k = 1,\dots,m$. Moreover

$$0 < \tau_1(x) < \dots < \tau_m(x) < b \text{ for all } x \in \mathbb{R}.$$

(H_4) For all $x \in \mathbb{R}$

$$\tau_k(I_k(x)) \leq \tau_k(x) < \tau_{k+1}(I_k(x)) \quad \text{for } k = 1,\dots,m.$$

(H_5) For all $a \in J$ fixed, $y \in B_b$ and for a.e. $t \in J$ we have

$$\tau_k'(y(t))\int_a^t (t-s)f(s, y_{\rho(t,y_t)})ds \neq 1 \quad \text{for } k = 1,\dots,m.$$

(H_6) The functions $I_k,\ \overline{I}_k,\ k = 1,2,\dots,m$ are continuous.

The next result is consequence of the phase space axioms.

Lemma III.1. *If $y : (-\infty,b] \to \mathbb{R}$ is a function such that $y_0 = \phi$ and $y|_J \in PC(J,\mathbb{R})$, then*

$$\|y_s\|_{\mathcal{B}} \leq (M_b + L^{\phi})\|\phi\|_{\mathcal{B}} + K_b \sup\{\|y(\theta)\|;\, \theta \in [0,\, \max\{0,s\}]\} \quad , \quad s \in \mathcal{R}(\rho^-) \cup J,$$

where

$$L^{\phi} = \sup_{t \in \mathcal{R}(\rho^-)} L^{\phi}(t).$$

Remark III.1. *We remark that condition (H_ϕ) is satisfied by functions which are continuous and bounded. In fact, if the space $\mathcal{B}$ satisfies axiom C_2 in [19] then there exists a constant $L > 0$ such that $\|\phi\|_{\mathcal{B}} \leq L\sup\{\|\phi(\theta)\| : \theta \in [-\infty,0]\}$ for every $\phi \in \mathcal{B}$ that is continuous and bounded (see [19] Proposition 7.1.1) for details. Consequently,*

$$\|\phi_t\|_{\mathcal{B}} \leq L\frac{\sup_{\theta\leq 0}\|\phi(\theta)\|}{\|\phi\|_{\mathcal{B}}}\|\phi\|_{\mathcal{B}},\ \textit{for every } \phi \in \mathcal{B}\backslash\{0\}.$$

Theorem III.1. *Assume that hypotheses (H_ϕ), (H_1)-(H_6). Then the problem (1)-(6) has at least one solution on $(-\infty,b]$.*

Proof. The proof will be given in a couple of steps.
Step 1: Consider the initial value problem

$$y''(t) = f(t, y_{\rho(t,y_t)}), \quad a.e.\ t \in J, \tag{7}$$
$$y(t) = \phi(t), \quad t \in (-\infty,0], \tag{8}$$
$$y'(0) = \eta. \tag{9}$$

Set

$$\tilde{C} = \{y : (-\infty,b] : y|_{(-\infty,0]} \in \mathcal{B} \text{ and } y \in C(J,\mathbb{R})\}.$$

Define the operator $N : \tilde{C} \to \tilde{C}$ by:

$$N(y)(t) = \begin{cases} \phi(t), \\ \quad \text{if} \quad t \in (-\infty,0], \\ \phi(0) + t\eta + \displaystyle\int_0^t (t-s)f(s, y_{\rho(s,y_s)})ds \\ \quad \text{if} \quad t \in [0,b]. \end{cases}$$

Clearly the fixed point of N are solutions to (7)–(9). Let $x(.) : (-\infty,b] \to \mathbb{R}$ be the function defined by:

$$x(t) = \begin{cases} \phi(t), & \text{if } \ t \in (-\infty,0], \\ \phi(0) + t\eta, & \text{if } \ t \in [0,b]. \end{cases}$$

Then $x_0 = \phi$. For each $z \in \mathcal{B}_b$ with $z_0 = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} 0, & \text{if } \ t \in (-\infty,0], \\ z(t), & \text{if } \ t \in [0,b]. \end{cases}$$

If $y(\cdot)$ satisfies the integral equation

$$y(t) = \phi(0) + t\eta + \int_0^t (t-s)f(s, y_{\rho(s,y_s)})ds.$$

We can decompose $y(.)$ into $y(t) = \overline{z}(t) + x(t)$, $0 \leq t \leq b$, which implies $y_t = \overline{z}_t + x_t$, for every $t \in [0,b]$, and the function $z(\cdot)$ satisfies

$$z(t) = \int_0^t (t-s)f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})ds.$$

Let

$$\mathcal{C}_0 = \{z \in \tilde{C} : z_0 = 0\}.$$

Let $\|.\|_0$ be the norm in $\mathcal{C}_0$ defined by

$$\begin{aligned}\|z\|_0 &= \|z_0\|_{\mathcal{B}} + \sup\{|z(s)| : 0 \leq s \leq b\} \\ &= \sup\{|z(s)| : 0 \leq s \leq b\} = \|z\|_b.\end{aligned}$$

We define the operator $P : \mathcal{C}_0 \to \mathcal{C}_0$ by

$$P(z)(t) = \int_0^t (t-s)f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})ds.$$

Obviously the operator N has a fixed point is equivalent to P has one, so we need to prove that P has a fixed point. We shall show that P satisfies the assumptions of Leray-Schauder alternative. The proof will be given in several Claims.

Claim 1: *P is continuous.*

Let $\{y^n\}$ be a sequence such that $y_0^n = 0$ and $y^n \to y$ in $\mathcal{C}_0$. Then for each $t \in J$,

$$\begin{aligned}|(Py^n)(t) - (Py)(t)| &\leq \int_0^t |(t-s)f(s, y^n_{\rho(s,y^n(s)})\\ &\quad - f(s, y_{\rho(s,y_s)})|ds\\ &\leq \int_0^b |t-s||f(s, y^n_{\rho(s,y_q(s)})\\ &\quad - f(s, y_{\rho(s,y_s)})|.\end{aligned}$$

Since f is Carathéodory we have $f(s, y^n_{\rho(s,y_q(s))}) \to f(s, y_{\rho(s,y_s)})$ as $n \to \infty$, for every $s \in J$. Now a standard application of the Lebesgue dominated convergence theorem implies that

$$\|Py^n - Py\|_0 \to 0 \quad \text{as} \quad n \to \infty.$$

and then P is continuous.

Claim 2: *P maps bounded set into a bounded set of $\mathcal{C}_0$.*

Indeed it is enough to show that for any $q > 0$, there exists a positive constant ℓ such that for each $z \in B_q = \{z \in \mathcal{C}_0 : \|z\|_0 \leq q\}$, one has $\|P(z)\|_0 \leq \ell$.
Let $z \in B_q$ by (H_2) we have for each $t \in J$,

$$\begin{aligned}|P(z)(t)| &\leq \int_0^t |t-s|\|f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)}\\ &\quad + x_{\rho(s,\overline{z}_s+x_s)})\|ds\\ &\leq \int_0^t |t-s|p(s)\psi(\|\overline{z}_{\rho(s,\overline{z}_s+x_s)}\\ &\quad + x_{\rho(s,\overline{z}_s+x_s)}\|)ds\\ &\leq \psi(K_bq\\ &\quad + K_b|\phi(0)| + M_b\|\phi\|_{\mathcal{B}})) \int_0^t |t-s|p(s)ds\\ &\leq \psi(K_bq + K_b|\phi(0)|\\ &\quad + M_b\|\phi\|_{\mathcal{B}})) \int_0^b |t-s|p(s)ds\\ &= l.\end{aligned}$$

Claim 3: *P maps bounded sets into equicontinuous sets of $\mathcal{C}_0$.*

Let $\tau_1, \tau_2 \in [0, b]$, $\tau_1 < \tau_2$, let B_q a bounded set of $\mathcal{C}_0$ as in Claim 2, and let $z \in B_q$,
then

$$\begin{aligned}&|(Pz)(\tau_2) - (Pz)(\tau_1)| \leq |\int_0^{\tau_2} (\tau_2 - s)f(s, y_{\rho(s,y_s)})ds\\ &- \int_0^{\tau_1} (\tau_2 - s)f(s, y_{\rho(s,y_s)})ds|\\ &\leq |\int_0^{\tau_2} (\tau_2 - s)f(s, y_{\rho(s,y_s)})ds\\ &- \int_0^{\tau_2} (\tau_1 - s)f(s, y_{\rho(s,y_s)})ds|\\ &+ |\int_0^{\tau_2} (\tau_1 - s)f(s, y_{\rho(s,y_s)})ds\\ &- \int_0^{\tau_1} (\tau_1 - s)f(s, y_{\rho(s,y_s)})ds|\\ &\leq |\int_0^{\tau_2} (\tau_2 - \tau_1)f(s, y_{\rho(s,y_s)})ds|\\ &+ |\int_{\tau_1}^{\tau_2} (\tau_1 - s)f(s, y_{\rho(s,y_s)})ds|\\ &\leq \psi(\|y_{\rho(s,y_s)}\|)(|\int_0^{\tau_2} p(s)(\tau_2 - \tau_1)ds|\\ &+ |\int_{\tau_1}^{\tau_2} p(s)(\tau_1 - s)ds|)\\ &\leq \psi(q^*)(|\int_0^{\tau_2} p(s)(\tau_2 - \tau_1)ds|\\ &+ |\int_{\tau_1}^{\tau_2} p(s)(\tau_1 - s)ds|).\end{aligned}$$

Where

$$q^* = K_bq + K_b|\phi(0)| + M_b\|\phi\|_{\mathcal{B}}$$

We see that $|(Pz)(\tau_2) - (Pz)(\tau_1)|$ tend to zero independently of $z \in B_q$ as $\tau_2 \to \tau_1$. As a consequence of claims 1 to 3 together with the Ascoli-Arzela theorem we can conclude that P is continuous and completely continuous.

Claim 4: *A priori bounds.*

Now it remains to show that the set

$$\mathcal{E} = \{z \in \mathcal{C}_0 : z = \lambda P(z) \quad \text{for some} \quad 0 < \lambda < 1\}$$

is bounded. Let $z \in \mathcal{E}$, then $z = \lambda P(z)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$,

$$z(t) = \lambda \int_0^t f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})ds.$$

Then for each $t \in J$, we have

$$|z(t)| \leq \lambda \int_0^t p(s)\psi(\|\overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}})ds, \tag{10}$$

but

$$\|\overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}} \leq \|\overline{z}_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}}$$

$$+\|x_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}}$$

$$\leq K(t)\sup\{|z(s)|:0\leq s\leq t\} + M(t)\|z_0\|_{\mathcal{B}} + K(t)\sup\{|x(s)| : 0\leq s\leq t\} + M(t)\|x_0\|_{\mathcal{B}}$$

$$\leq K_b\sup\{|z(s)|:0\leq s\leq t\} + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)|,$$

and then

$$\|\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}} \leq K_b\sup\{|z(s)| \quad (11)$$
$$: 0\leq s\leq t\} + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)|. \quad (12)$$

If we name $w(t)$ the right hand side of (11), then we have

$$\||\overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}} \leq w(t).$$

Therefore (10) becomes

$$|z(t)| \leq \int_0^t p(s)\psi(w(s))ds. \quad (13)$$

Using (13) in the definition of w, we have

$$w(t) \leq K_b\int_0^t p(s)\psi(w(s))ds + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)|.$$

Let us take the right hand-side of the last inequality as $v(t)$. Then we have

$$w(t)\leq v(t) \quad \text{for all } \ t\in J,$$

$$v(0) = K_b|\phi(0)| + M_b\|\phi\|_{\mathcal{B}},$$

and

$$v'(t) = K_b p(t)\psi(w(t)), \quad a.e. \ \ t\in J.$$

Using the nondecreasing character of ψ we get

$$v'(t) \leq K_b p(t)\psi(v(t)), \quad a.e. \ \ t\in J.$$

That is

$$\frac{v'(t)}{\psi(v(t))} \leq K_b p\,(t)\,, \quad a.e. \ \ t\in J.$$

Integrating from 0 to t we get

$$\int_0^t \frac{v'(s)}{\psi(v(s))}ds \leq K_b\int_0^t p\,(s)\,ds.$$

By a change of variable and using $(\mathcal{H}_2)$ we get

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq K_b\int_0^b p\,(s)\,ds < \int_C^\infty \frac{du}{\psi(u)}.$$

Hence there exists a constant K_* such that

$$v(t)\leq K_* \quad \text{for all } \ t\in J.$$

and hence $\|\overline{z}_t + x_t\|_{\mathcal{C}} \leq w(t) \leq K_*,\ t\in J$. From (13) we have that

$$\|z\|_0 \leq \psi(K_*)\int_0^b p(s)ds := K_1.$$

Set

$$U = \{y\in\mathcal{C}_0 : \sup\{|z(t)|,\ 0\leq t\leq b\} \leq K_1+1.$$

From the choice of U, there is no $y\in\partial U$ such that $y=\lambda P(y)$ for some $\lambda\in[0,1]$. The nonlinear alternative of Leray-Schauder type [14] implies that P has a fixed point, hence N has a fixed point which is a solution of problem (7)-(9). Denote this solution by y_1.

Define the function

$$r_{k,1}(t) = \tau_k(y_1(t)) - t \text{ for } \ t\geq 0.$$

(H_3) implies that

$$r_{k,1}(0)\neq 0 \text{ for } \ k=1,\ldots,m.$$

If

$$r_{k,1}(t)\neq 0 \text{ on } J \text{ for } k=1,\ldots,m,$$

i.e

$$t\neq\tau_k(y_1(t)) \text{ on } J \text{ for } k=1,\ldots,m,$$

then y_1 is solution of the problem (1)-(6).

Now we consider the case when $r_{1,1}(t)=0$ for some $t\in J$. Since $r_{1,1}(0)\neq 0$ and $r_{1,1}$ is continuous, there exists $t_1>0$ such that

$$r_{1,1}(t_1)=0 \text{ and } r_{1,1}(t)\neq 0 \text{ for all } t\in[0,t_1).$$

Thus by (H_3) we have

$$r_{k,1}(t)\neq 0 \text{ for all } t\in[0,t_1) \text{ and } k=1,\ldots,m.$$

Step 2: Consider the following problem

$$y''(t) = f(t, y_{\rho(t,y_t)}), \text{ for a.e., } \ t\in[t_1,b] \quad (14)$$

$$y(t_1^+) = I_1(y_1(t_1^-)) \quad (15)$$

$$y'(t_1^+) = \overline{I}_1(y_1(t_1^-)), \quad (16)$$

$$y(t) = y_*(t),\ t\in(-\infty,t_1]. \quad (17)$$

Where

$$y_*(t) = \begin{cases} y_1(t), & \text{if } \ t\in[0,t_1] \\ \phi(t), & \text{if } \ t\in(-\infty,0] \end{cases}$$

Let

$$\mathcal{C}_1 = \{y \in \mathcal{C}((t_1, b], \mathbb{R}), y(t_1^+) \ \text{ exist}\},$$

and

$$\mathcal{C}_* = \{y : (-\infty, b] \to \mathbb{R} : y \in C((-\infty, t_1], \mathbb{R}) \cap \mathcal{C}_1\}.$$

Consider the operator $N_1 : \mathcal{C}_* \to \mathcal{C}_*$ defined by:

$$N(y)(t) = \begin{cases} y_*(t), \\ \text{ if } \quad t \in (-\infty, t_1], \\ I_1(y_1(t_1)) + (t - t_1)\left(\overline{I}_1(y_1(t_1))\right) \\ \text{ if } \ t \in [t_1, b] \\ + \displaystyle\int_{t_1}^{t} (t-s) f(s, y_{\rho(s,y_s)}) ds. \end{cases}$$

Let $x(.) : (-\infty, b] \to \mathbb{R}$ be the function defined by

$$x(t) = \begin{cases} I_1 \qquad (y_1(t_1)) + (t - t_1)\left(\overline{I}_1(y_1(t_1))\right), \\ \qquad\qquad \text{ if } \ t \in (t_1, b], \\ y_*(t), \\ \qquad\qquad \text{ if } \ t \in (-\infty, t_1]. \end{cases}$$

Then $x_{t_1} = y_1$. For each $z \in \mathcal{C}_*$ with $z_{t_1} = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} 0, & \text{if } \ t \in (-\infty, t_1], \\ z(t), & \text{if } \ t \in [t_1, b]. \end{cases}$$

If $y(\cdot)$ satisfies the integral equation

$$\begin{aligned} y(t) = & I_1(y_1(t_1)) + (t - t_1)\left(\overline{I}_1(y_1(t_1))\right) \\ & + \int_{t_1}^{t} (t-s) f(s, y_{\rho(s,y_s)}) ds. \end{aligned}$$

We can decompose $y(.)$ into $y(t) = \overline{z}(t) + x(t)$, $t_1 \le t \le b$, which implies $y_t = \overline{z}_t + x_t$, for every $t \in [t_1, b]$, and the function $z(\cdot)$ satisfies

$$z(t) = \int_{t_1}^{t} (t-s) f(s, \overline{z}_{\rho(s,\overline{z}_s + x_s)} + x_{\rho(s,\overline{z}_s + x_s)}) ds.$$

Let

$$\mathcal{C}_{t_1} = \{z \in \mathcal{C}_*, z(t_1) = 0\}.$$

Let the operator $P : \mathcal{C}_{t_1} \to \mathcal{C}_{t_1}$ by

$$P_1(z)(t) = \int_{t_1}^{t} (t-s) f(s, \overline{z}_{\rho(s,\overline{z}_s + x_s)} + x_{\rho(s,\overline{z}_s + x_s)}) ds.$$

As in Step 1 we can show that P_1 is continuous and completely continuous and if z is a solution for the equation $z = \lambda P_1(z)$, for some $\lambda \in (0, 1)$ there exist $K_{1_*} > 0$ such that

$$\|z\|_\infty \le K_{1_*} > 0.$$

Set

$$U_1 = \{y \in \mathcal{C}_{t_1} : \sup\{\|z(t)\| : t_1 \le t \le b\} \le K_{1_*} + 1.$$

As a consequence of Leray-Schauder's nonlinear alternative type we deduce that P has a fixed point z in U_1. Thus N_1 has a fixed point y which is a solution of problem (14)-(17), denote this solution by y_2.

Define

$$r_{k,2}(t) = \tau_k(y_2(t)) - t \text{ for } \ t \ge t_1.$$

If

$$r_{k,2}(t) \ne 0 \text{ on } (t_1, b] \text{ for } k = 1, \ldots, m,$$

then

$$y(t) = \begin{cases} y_1(t), & \text{if } t \in [0, t_1], \\ y_2(t), & \text{if } t \in (t_1, b], \end{cases}$$

is solution of the problem (1)-(6). It remains to consider the case when

$$r_{2,2}(t) = 0, \text{ for some } t \in (t_1, b].$$

By (H_4) we have

$$\begin{aligned} r_{2,2}(t_1^+) &= \tau_2(y_2(t_1^+)) - t_1 \\ &= \tau_2(I_1(y_1(t_1^-))) - t_1 \\ &> \tau_1(y_1(t_1^-)) - t_1 \\ &= r_{1,1}(t_1) = 0. \end{aligned}$$

Since $r_{2,2}$ is continuous, there exists $t_2 > t_1$ such that $r_{2,2}(t_2) = 0$ and $r_{2,2}(t) \ne 0$ for all $t \in (t_1, t_2)$. By (H_3) we have :

$$r_{k,2} \ne 0 \text{ for all } t \in (t_1, t_2) \text{ and } k = 2, \ldots, m.$$

Suppose now that there exists $\overline{s} \in (t_1, t_2]$ such that $r_{1,1}(\overline{s}) = 0$. From (H_4) it follows that

$$\begin{aligned} r_{1,2}(t_1^+) &= \tau_1(y_2(t_1^+)) - t_1 \\ &= \tau_1(I_1(y_1(t_1^-))) - t_1 \\ &\le \tau_1(y_1(t_1)) - t_1 \\ &= r_{1,1}(t_1) = 0. \end{aligned}$$

Thus the function $r_{1,2}$ attains a nonnegative maximum at some point $s_1 \in (t_1, b]$. Since

$$y_2'(t) = \int_{t_1}^{t} f(s, y_{2_{\rho(s,y_{2_s})}}) ds$$

then

$$r_{1,2}'(s_1) = \tau_1'(y_2(s_1) y_2'(s_1) - 1 = 0.$$

Therefore

$$\tau_1'(y_2(s_1)) \int_{t_1}^{s_1} f(s, y_{2_{\rho(s,y_{2_s})}}) ds = 1,$$

which contradicts (H_5)

Step 3: We continue this process and taking into account that $y_{m+1} := y|_{[t_m,b]}$ is a solution to the problem

$$y''(t) = f(t, y_{\rho(t,y_t)}), \text{ for a.e., } t \in (t_m, b], \quad (18)$$

$$y(t_m^+) = I_m(y_m(t_m^-)), \quad (19)$$

$$y'(t_m^+) = \overline{I}_m(y_m(t_m^-)). \quad (20)$$

The solution y of the problem (1)-(6) is then defined by

$$y(t) = \begin{cases} y_1(t), & \text{if } t \in (-\infty, t_1], \\ y_2(t), & \text{if } t \in (t_1, b], \\ \dots \\ y_{m+1}(t), & \text{if } t \in (t_m, b]. \end{cases}$$

IV. Example

To apply our results, we consider the functional differential equation with variables times and state-dependent delay of the form :

$$y''(t) = \frac{(y(t-\sigma(y(t))))^2}{(t^2+1)(t+2)(1+(y(t-\sigma(y(t))))^2)} a.e, \quad (21)$$

$$t \in [0,1], t \neq \tau_k(y(t)), k = 1, \dots, m, \quad (22)$$

$$\Delta y|_{t=\tau_k(y(t))} = I_k(y(t)), t = \tau_k(y(t)), k = 1, \dots, m, \quad (23)$$

$$\Delta y'|_{t=\tau_k(y(t))} = \overline{I}_k(y(t)), t = \tau_k(y(t)), k = 1, \dots, m, \quad (24)$$

$$y(t) = \phi(t), t \in (-\infty, 0], \quad (25)$$

$$y'(0) = \eta, \quad (26)$$

where $\sigma \in C(\mathbb{R}, [0,\infty))$. Set $\gamma > 0$. For the phase space, we choose $\mathcal{B}$ to be defined by

$$\mathcal{B}_\gamma = \{y \in PC((-\infty,0],\mathbb{R}) : \lim_{\theta\to-\infty} e^{\gamma\theta} y(\theta) \text{ exists}\}$$

with the norm

$$\|y\|_\gamma = \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}|y(\theta)|,$$

where

$$PC((-\infty,0],\mathbb{R}) = \Big\{ y : (-\infty,0] \to \mathbb{R} : y \text{ is continuous at } t \neq \tilde{t}_k,\ y(\tilde{t}_k^-) = y(\tilde{t}_k) \text{ and } y(\tilde{t}_k^+) \text{ exists for all } k = 1, \dots, m \Big\}.$$

Let $y : (-\infty, b] \to \mathbb{R}$ be such that $y_0 \in \mathcal{B}_\gamma$. Then

$$\begin{aligned} \lim_{\theta\to-\infty} e^{\gamma\theta} y(\theta) &= \lim_{\theta\to-\infty} e^{\gamma\theta} y(t+\theta) \\ &= \lim_{\theta\to-\infty} e^{\gamma(\theta-t)} y(\theta) \\ &= e^{\gamma\ t} \lim_{\theta\to-\infty} e^{\gamma\theta} y_0(\theta). \end{aligned}$$

Hence $y_t \in \mathcal{B}_\gamma$. Finally we prove that

$$\|y_t\|_\gamma \leq K(t)\sup\{\ |y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_\gamma,$$

where $K = M = H = 1$. We have $y(t) = y(t+\phi)$. If $t + \theta \leq 0$ we get

$$\|y_t(\theta)\| \leq \sup\{\ |y(s)| : -\infty \leq s \leq 0\}.$$

For $t + \theta \geq 0$ we have

$$\|y_t(\theta)\| \leq \sup\{\ |y(s)| : 0 \leq s \leq t\}.$$

Thus for all $t + \theta \in [0,1]$, we get

$$\|y_t(\theta)\| \leq \sup\{\ |y(s)| : -\infty \leq s \leq 0\} + \sup\{\ |y(s)| : 0 \leq s \leq t\}.$$

Thus

$$\|y_t\|_\gamma \leq \|y\|_0 + \sup\{\ |y(s)| : 0 \leq s \leq t\}.$$

It is clear that $(\mathcal{B}_\gamma, \|y\|_\gamma)$ is a Banach space. We can conclude that $\mathcal{B}_\gamma$ is a phase space.

Set

$$f(t,u) = \frac{(u(0))^2}{(t^2+1)(t+2)(1+(u(0)^2)}, \quad (t,u) \in [0,1] \times \mathcal{B}_\gamma,$$

$$\rho(t,u) = t - \sigma(u(0)), \quad (t,u) \in [0,1] \times \mathcal{B},$$

$$\tau_k(x) = 2k - \frac{1}{2^{k+1}(1+x^2)},$$

$$I_k(x) = d_k x,$$

$$\overline{I_k}(x) = \overline{d_k} x.$$

From the the definition of τ_k we have $\tau_k(x) \neq 0$ and

$$\tau_{k+1}(x) - \tau_k(x) = 2 + \frac{1}{2^{k+2}(1+x^2)} > 0 \text{ for all } x \in \mathbb{R} \text{ and } k = 1, \dots, m.$$

So

$$0 < \tau_1(x) < \tau_2(x) < \tau_3(x) < \dots < \tau_k(x) \text{ for all } x \in \mathbb{R}.$$

Also

$$\tau_k(I_k(x)) - \tau_k(x) = \frac{(b_k^2-1)x^2}{2^{k+1}(1+x^2)(1+b_k^2x^2)} \leq 0$$

and

$$\tau_{k+1}(I_k(x)) - \tau_k(x) = \frac{2^{k+3}(1+x^2)(1+b_k^2x^2)+1+(2b_k^2-1)x^2}{2^{k+1}(1+x^2)(1+b_k^2x^2)} > 0$$

for all $x \in \mathbb{R}$ and $k = 1, \ldots, m$. Thus

$$\tau_k(I_k(x)) \leq \tau_k(x) \leq \tau_{k+1}(I_k(x)) \text{ for all } x \in \mathbb{R} \text{ and } k = 1, \ldots, m.$$

We can easily show that

$$\begin{aligned} |\tau_k'(x)\int_a^t f(s, y_{\rho(s,y_s)})ds| &\leq |\tau_k'(x) \\ &\cdot |\int_a^t |f(s, y_{\rho(s,y_s)})|ds \\ &\leq \frac{1}{2}|t-a| \\ &< 1. \end{aligned}$$

Assume that $p(t) = \frac{1}{t^2+1}$ and $\psi(x) = 1$.Then

$$|f(t,u)| \leq \frac{1}{t^2+1}\psi(\|u\|_b) \text{ for all } (t,u) \in [0,1] \times \mathcal{B}_b$$

$$\int_c^{+\infty} \frac{du}{\psi(u)} = +\infty$$

It is clear that all conditions of Theorem III.1 are satisfied. Hence problem (20)-(26) has at least one solution defined on $]-\infty, b]$.

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Pseudo-differential operators and applications to Partial differential equations

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Abstract: **The aim of this work is to study pseudo-differential operators. We define this operator's class and give an application to the theory of partial differential equations.**

Keyword: **Pseudo-differential operators, symbols, elliptic PDO.**

I. Introduction

The theory of pseudo-differential operators (which in what follows will be abbreviated as PDO) is a tool to solve elliptic partial differential equations. It is developed in the middle of sixty thank's to works of Hörmander, Nirenberg and others. This theory has a great role in mathematics and mathematics-physic like microlocal analysis and quantum physics.

The general form of pseudo-differential operator I is

$$(Iu)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i<x-y,\xi>} a(x,y,\xi) u(y) dy\, d\xi, \tag{1}$$

where a is a symbol and $u \in C_0^\infty(\mathbb{R}^n)$ (the space of infinitely differentiable functions with compact support).

A linear partial differential operator P of order m is an application in $C^\infty(\mathbb{R}^n)$

$$(Pu)(x) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha u(x),$$

with $a_\alpha \in C^\infty(\mathbb{R}^n)$ are the coefficients of the operator P.

The Fourier transform of an application u is defined by

$$\widehat{u}(\xi) = (\mathcal{F}u)(\xi) = \int_{\mathbb{R}^n} e^{-i<x,\xi>} u(x) dx$$

is an isomorphism in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ with inverse given by

$$(\mathcal{F}^{-1}v)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i<x,\xi>} v(\xi) d\xi.$$

The Fourier transform of an application converts differential problems to algebraic problems, one of its elementary properties is

$$\widehat{D^\alpha u}(\xi) = \xi^\alpha \widehat{u}(\xi)$$

From the previous formulas we deduce

$$\begin{aligned} D^\alpha u(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i<x,\xi>} \xi^\alpha \widehat{u}(\xi) d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i<x-y,\xi>} \xi^\alpha u(y) dy\, d\xi \end{aligned}$$

So we can write P as

$$(Pu)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i<x-y,\xi>} p(x,\xi) u(y) dy\, d\xi,$$

where p is the *symbol* of the differential operator P defined by

$$p(x,\xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha,$$

is a polynomial of degree m in ξ with coefficients depending in x.

So we have represented a differential operator by its symbol using Fourier transform and its basic properties. We can extend this representation by considering a new class of symbols that are not polynomials in ξ. This new operators are called "Pseudo-differential operators".

II. PSEUDO-DIFFERENTIAL OPERATORS

A. *Notions on PDO*

In the sequel, we assume X an open set in $\mathbb{R}^n$. A PDO is defined by its symbol.

***Definition** 2.1: Let $m, \rho, \delta \in \mathbb{R}$ with $0 \leq \rho, \delta \leq 1$. A symbol is a function $a \in C^\infty(X \times \mathbb{R}^n)$ such that for any compact $K \subset X$ and any multi-indices α and β, there exists a constant $C_{\alpha,\beta,K}$ for which*

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leq C_{\alpha,\beta,K}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|},$$

where $x \in K$, $\xi \in \mathbb{R}^n$.

We note $S^m_{\rho,\delta}(X \times \mathbb{R}^n)$ the set of symbols of order am and type ρ, δ and $S^{-\infty}_{\rho,\delta}(X \times \mathbb{R}^n) = \bigcap_m S^m_{\rho,\delta}(X \times \mathbb{R}^n)$.

***Example** 2.1:* If $a \in C^\infty$ and a is a homogeneous function of degree m with respect to ξ for large $|\xi|$, then a is a symbol of degree m and type $1, 0$.

To facilitate calculus on symbols we mention some elementary properties.

***Proposition** 2.1:*
(i) *If $a \in S^m_{\rho,\delta}$ then $\partial_x^\alpha \partial_\xi^\beta a \in S^{m-\rho|\alpha|+\delta|\beta|}$;*
(ii) *If $m \leq l$ then $S^m_{\rho,\delta} \subset S^l_{\rho,\delta}$;*
(iii) *If $a \in S^m_{\rho,\delta}$ et $b \in S^l_{\rho,\delta}$, then $ab \in S^{m+l}$.*

***Definition** 2.2: The integral*

$$(Iu)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i<x-y,\xi>} a(x,y,\xi) u(y) dy\, d\xi,$$

where $a \in S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n)$ and $u \in C_0^\infty(\mathbb{R}^n)$ is said pseudo-differential operator.

The set of PDO is denoted $\Psi^m_{\rho,\delta}(X)$.

***Definition** 2.3: The distribution $K \in \mathcal{D}'(X \times X)$ defined by*

$$K(x,y) = \int e^{i(x-y)\xi}\, a(x,y,\xi)\, d\xi \qquad (2)$$

is called kernel of the PDO I.

***Example** 2.2:* A linear partial differential operator of order m

$$A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

where $a_\alpha \in C^\infty$ defines a PDO of order m and type $1, 0$, his symbol is given by

$$\sigma_A(x,\xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha.$$

To give a meaning to the previous integral we introduce the notion of phase function and oscillatory integrals.

***Definition** 2.4: We call $\phi(x,\xi)$ a phase function if $\phi(x,\xi) \in C^\infty(X, \mathbb{R}^n \setminus 0)$, $\phi(x,\xi)$ is real valued and positively homogeneous of degree 1 in ξ (i.e. $\phi(x,t\xi) = t\phi(x,\xi)$ for any $x \in X$, $\xi \in \mathbb{R}^n \setminus 0$ and $t > 0$) and $\phi(x,\xi)$ does not have critical points for $\xi \neq 0$.*

***Definition** 2.5: The integral*

$$(Iu)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi) u(x)\, dx\, d\xi$$

in which $a(x,\xi) \in S^m_{\rho,\delta}(X \times \mathbb{R}^n)$ and $\phi(x,\xi)$ is a phase function is called an oscillatory integral.

The integral just defined converges if $m < -n$. In the other case we use the technique of oscillatory integrals developed by Hörmander which consists to construct a differential operator L that satisfies

$${}^tL e^{i\phi(x,\xi)} = e^{i\phi(x,\xi)},$$

where ${}^tL$ is the transpose of the differential operator L. Using this fact, an integration by parts k times shows that

$$(Iu)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} L^k(a(x,\xi)u(x))\, dx\, d\xi \qquad (3)$$

Putting $s = \min(\rho, 1-\delta)$, we deduce $L^k(au) \in S^{m-ks}_{\rho,\delta}(X \times \mathbb{R}^n)$. If $\rho > 0$ and $\delta < 1$ (so that $s > 0$), which will always be assumed in the sequel, then the formula (3) allows us to define the integral I for an arbitrary m if we select k so that $m - ks < -n$. This makes the integral (3) absolutely convergent.

***Proposition** 2.2: A is a linear bounded operator from $C_0^\infty(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$.*

We study now the algebraic structure of the set of PDO. To this end if we want to calculate the product (composition) of two PDO A and B, one has to require that the image of the operator B is compact, if we consider $A \circ B$, but in general we have not this case, so we have to introduce another class of pseudo-differential operators known as " Properly supported pseudo-differential operators".

B. The Algebra of Pseudo-differential operators

Let I be a PDO with kernel K_I and let $supp_{K_I}$ denotes the support of K_I. Recall that a continuous map $f : X \to Y$ between topological spaces X and Y is called *proper* if for any compact $K \subset Y$, the inverse image $f^{-1}(K)$ is a compact in X.

***Definition** 2.6: A PDO I is called properly supported if both canonical projections $\Pi_1, \Pi_2 : supp_{K_I} \to X$ are proper maps.*

So we can state the following theorem.

***Theorem** 2.1: Let I be a properly supported PDO. Then I defines a map*

$$I : C_0^\infty(X) \to C_0^\infty(X).$$

To give some combination between classes of PDO we give the important theorem.

***Theorem** 2.2: Any pseudo-differential operator I can be written in the form $I = I_0 + I_1$ where I_0 is a properly supported PDO and I_1 has kernel $K_{I_1} \in C^\infty(X \times X)$.*

We return now to symbol calculus by introducing the symbol of properly supported PDO.

***Definition** 2.7: Let I be a properly supported PDO. Its symbol is a function $\sigma_I(x, \xi)$ on $X \times \mathbb{R}^n$ defined by*

$$\sigma_I(x, \xi) = e_{-\xi}(x)\, I\, e_\xi(x),$$

where $e_\xi(x) = e^{ix\xi}$.

Since $e^{ix\xi}$ is an infinitely differentiable function of ξ with values in $C^\infty(X)$ and I is a continuous linear operator on $C^\infty(X)$, it is clear that $\sigma_I(x, \xi)$ is also an infinitely differentiable function of ξ taking values in $C^\infty(X)$, therefore $\sigma_I(x, \xi) \in C^\infty(X \times \mathbb{R}^n)$.

*1) **An expression for the symbol of a properly supported PDO**:* In this and the following paragraphs we assume $\delta < \rho$.

***Theorem** 2.3:* Let I be a properly supported PDO and $\sigma_I(x, \xi)$ its symbol. Then

$$\sigma_I(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha\, D_y^\alpha\, a(x, y, \xi)|_{y=x}.$$

*2) **The symbol of the transposed operator**:* The transposed operator ${}^tI$ is defined by

$$< Iu, v >=< u,^t Iv >$$

for any $u, v \in C_0^\infty(X)$, where

$$< u, v >= \int u(x)v(x)dx.$$

Therefore, if $I \in \Psi^m_{\rho,\delta}(X)$ where $a \in S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n)$, the transpose ${}^tI$ is given by

$${}^tIv(y) = \iint e^{i(x-y)\xi}\, a(x, y, \xi)v(x)dx\, d\xi;$$

which with the change of variable $\eta = -\xi$ gives

$${}^tIv(y) = \iint e^{i(y-x)\xi}\, a(x, y, -\eta)v(x)dx\, d\eta.$$

Consequently, $t^I \in \Psi^m_{\rho,\delta}(X)$.

***Theorem** 2.4: Let I be a properly supported PDO with symbol $\sigma_I(x, \xi)$ and $\sigma'_I(x, \xi)$ the symbol of ${}^tI$, then*

$$\sigma'_I(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha\, D_x^\alpha \sigma_I(x, -\xi).$$

*3) **The composition formula**:*

***Theorem** 2.5: Let I_2 and I_2 be two properly supported PDO in X and let their symbols be $\sigma_{I_1}(x, \xi)$ and $\sigma_{I_2}(x, \xi)$ respectively. Then the composition BA is then a properly supported PDO, whose symbol satisfies the relation*

$$\sigma_{BA}(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_{I_2}(x, \xi) D_x^\alpha \sigma_{I_1}(x, \xi).$$

C. Boundedness and compactness of PDO

Operator's continuity is an important question often asked in the theory of operators. When the operator on question is linear, the notions of continuity and boundedness are equivalents. So we are interesting to study its boundedness between Banach spaces and particularly the famous Lebegue's spaces L^2. The importance of the space L^2 is arisen from its structure, it is complete, defines a Hilbert space and in addition is the most space used by physicists. So in this paragraph we treat L^2 boundedness and L^2 compactness of PDO.

***Theorem** 2.6: Let $I \in \Psi^0_{\rho,\delta}(\mathbb{R}^n)$, $0 \leq \delta < \rho \leq 1$ and $supp_{K_I}$ be compact in $\mathbb{R}^n \times \mathbb{R}^n$. Then*

$$||Iu|| \leq C||u||$$

where $C > 0$, and I can be extended to a linear continuous operator on $L^2(\mathbb{R}^n)$.

For the proof we need the following theorem.

***Theorem** 2.7: Let I be a properly supported PDO in $\Psi^0_{\rho,\delta}(\mathbb{R}^n)$, with $0 \leq \delta < \rho \leq 1$ and X an open set in $\mathbb{R}^n$. Suppose there exists a constant M such that*

$$\overline{\lim_{\substack{|\xi|\to\infty \\ x\in K}}} |\sigma_I(x,\xi)| < M,$$

for any compact set $K \subset X$. Then there exists a properly supported integral operator with hermitian kernel $R \in C^\infty(X \times X)$ such that

$$(Iu, Iu) \leq M^2(u,u) + (Ru,u),\ \forall u \in C_0^\infty(X).$$

If, in addition, $supp_{K_I}$ is compact in $X \times X$ then $supp\, R$ is also compact in $X \times X$.

We deduce the compactness theorem.

***Theorem** 2.8: Let $I \in \Psi^0_{\rho,\delta}(\mathbb{R}^n)$ with $0 \leq \delta < \rho \leq 1$ let $supp_{K_I}$ be compact in $\mathbb{R}^n \times \mathbb{R}^n$ and*

$$\sup_x |\sigma_I(x,\xi)| \to 0 \ \text{ as } |\xi| \to +\infty.$$

Then I extends to a compact operator in $L^2(\mathbb{R}^n)$.

III. Applications

The main question in the theory of partial differential equations is how to solve the equation

$$Lu = f$$

for a given partial differential operator L and a given function f . In other words, how to find the inverse of L, i.e. an operator L^{-1} such that

$$L \circ L^{-1} = L^{-1} \circ L = I \tag{4}$$

is the identity operator (on some space of functions where everything is well-defined). In this case function $u = L^{-1}f$ gives a solution to the partial differential equation $Lu = f$. First of all we can observe that if operator L is an operator with variables coefficients in most cases it is impossible or is very hard to find an explicit formula for its inverse L^{-1} (even when it exists). However, in many questions in the theory of partial differential equations one is actually not so much interested in having a precise explicit formula for L^{-1}. Indeed, in reality one is mostly interested not in knowing the solution u to the equation $Lu = f$ explicitly but rather in knowing some fundamental properties of u. Thus, the question becomes whether we can say something about singularities of u knowing singularities of $f = Lu$. In this case we do not need to solve equation $Lu = f$ exactly but it is sufficient to know its solution modulo the class of smooth functions. Namely, instead of L^{-1} in (4) one is interested in finding an "approximate" inverse of L modulo smooth functions, i.e. an operator P such that

$$u = Pf$$

solves equation $Lu = f$ modulo smooth functions, i.e. if $(PL - I)f$ and $(LP - I)f$ are smooth for all functions f from some class.

***Definition** 3.1: A symbol $a \in S^m_{1,0}(X \times \mathbb{R}^n)$, $m \in \mathbb{R}$, is said to be elliptic if there exist $C, R > 0$ such that*

$$|a(x,\xi)| \geq C|\xi|^m, \qquad \forall|\xi| \geq R,\ x \in X.$$

***Definition** 3.2: A pseudo-differential operator is called elliptic if its symbol is elliptic.*

***Example** 3.1:* The symbol of the Laplacian operator $\Delta = \sum_{j=1}^{n} D_{x_j}^2$ is $-\sum_{j=1}^{n} \xi^2$, so it is elliptic.

***Definition** 3.3: Let $A \in \Psi_{1,0}^{m}(\mathbb{R}^n)$. A properly supported pseudo-differential operator P is called parametrix of A if it satisfies*

$$PA = I + R_l \text{ and } AP = I + R_r,$$

where $R_l, R_r \in \Psi_{1,0}^{-\infty}(\mathbb{R}^n)$.

***Theorem** 3.1: Any elliptic operator $A \in \Psi_{1,0}^{m}(\mathbb{R}^n)$ has a parametrix $P \in \Psi_{1,0}^{-m}(\mathbb{R}^n)$.*

IV. Fourier integral operators

Another family of operators intimately connected to the theory of PDO is known as Fourier integral operators which has the form

$$Fu(x) = \int e^{i\phi(x,\theta)} a(x,\theta) \mathcal{F}u(\theta)\, d\theta$$

where $u \in \mathcal{S}(\mathbb{R}^n)$, $a(x,\theta)$ is a symbol and $\phi(x,\theta)$ is a phase function.

So a PDO is a Fourier integral operator with phase function $< x, \theta >$.

This class of operators appear in the expression of the solutions of the hyperbolic partial differential equations and is a generalization.

V. Conclusion and Perspectives

In this article we defined PDO and studied most important properties and different calculus on its. We treated continuity in some classes of symbols by given some conditions on types of symbols. Our purpose is to generalize this results by considering another types of symbols and conditions on the phase functions in linear and multi-linear cases for both pseudo-differential and Fourier Integral operators.

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An Antiplane Electro-Elastic Problem with the Power-Law Friction

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Abstract— *In this paper the material used is electro-elastic and the friction and it is modeled with Tresca's law and the foundation is assumed to be electrically conductive. First we derive the well posedness mathematical model. In the second step, we give the classical variational formulation of the model which is given by a system coupling an evolutionary variational equality for the displacement field and a time-dependent variational equation for the potential field. Then we prove the existence of a unique weak solution to the model by using the Banach fixed-point Theorem.*

Keywords— *Tresca's friction, electro-elastic material, variational inequality, weak solution, fixed point, antiplane shears deformation.*

Mathematics Subject Classification— ***74G25, 49J40, 74F15, 74M10***

1. INTRODUCTION

We consider the antiplane contact problem for electro-elastic materials with Tresca friction law. In this new work, we assume that the dispalcement is parallel to the generators of the cylinder and is dependent of the axial coordinate. Our interest is to describe a physical process (for more details see [1, 4, 5, 6, 7, 8]) in which both antiplane shear, contact, state of material with Trescafriction law and piezoelectric effect are involved, leading to a well posedness mathematical problem. In the variational formulation, this kind of problem leads to an integro-differential inequality. The main result we provide concerns the existence of a unique weak solution to the model, see for instance [2, 3, 6] for details.

The rest of the paper is structured as follows. In Section s:2 we describe the well posedness mathematical model of the frictional contact process between electro-elastic body and a conductive deformable foundation. In Section s:3 we derive the variational formulation. It consists of a variational inequality for the displacement field coupled with a time-dependent variational equation for the electric potential. We state our main result, the existence of a unique weak solution to the model in Theorem 3.1. The Proof of the Theorem is provided in the end of Section s: 4, where it is based on arguments of evolutionary inequalities, and a fixed point Theorem.

2. THE MODEL

In this section, we consider a piezoelectric body $\mathcal{B}$ identified with a region in $\mathbb{R}^3$ it occupies in a fixed and undistorted reference configuration. We assume that $\mathcal{B}$ is a cylinder with generators parallel to the x_3 -axes with a cross-section which is a regular region Ω in the x_1 , x_2 -plane, $Ox_1x_2x_3$ being a Cartesian coordinate system. The cylinder is assumed to be sufficiently long so that the end effects in the axial direction are negligible. Thus, $\mathcal{B}=\Omega\times(-\infty,+\infty)$. The cylinder is acted upon by body forces of density $\mathbf{f}_0$ and has volume free electric charges of density q_0 . It is also constrained mechanically and electrically on the boundary. To describe the boundary conditions, we denote by $\partial\Omega=\Gamma$ the boundary of Ω and we assume a partition of Γ into three open disjoint parts Γ_1 , Γ_2 and Γ_3 , on the one hand, and a partition of $\Gamma_1\cup\Gamma_2$ into two open parts Γ_a and Γ_b , on the other hand. We assume that the one-dimensional measure of Γ_1 and Γ_a , denoted meas Γ_1 and meas Γ_a , are positive.

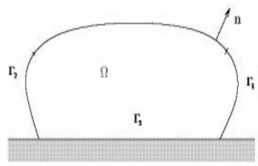

FIGURE 1. Deformable solid Ω on contact with a rigid foundation

The cylinder is clamped on $\Gamma_1\times(-\infty,+\infty)$ and therefore the displacement field vanishes there.

Surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2 \times (-\infty,+\infty)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (-\infty,+\infty)$ and a surface electrical charge of density q_2 is prescribed on $\Gamma_b \times (-\infty,+\infty)$. The cylinder is in contact over $\Gamma_3 \times (-\infty,+\infty)$ with a conductive obstacle, the so called foundation. The contact is frictional and is modeled with Tresca's law. We are interested in the deformation of the cylinder on the time interval $[0,T]$. We assume that

$$\mathbf{f}_0 = (0,0,\ f_0)\ with\ f_0 = f_0(x_1,\ x_2,t):\ \Omega\times[0,T]\to \mathrm{IR}, \quad (1)$$

$$\mathbf{f}_2 = (0,0,\ f_2)\ with\ f_2 = f_2(x_1,\ x_2,t):\ \Gamma\times[0,T]\ \to \mathrm{IR}, \quad (2)$$

$$q_0 = q_0(x_1,\ x_2,t)\ :\ \Omega\times[0,T]\to \mathrm{IR}, \quad (3)$$

$$q_2 = q_2(x_1,\ x_2,t):\ \Gamma_b\times[0,T]\to \mathrm{IR}. \quad (4)$$

The forces (1), (2) and the electric charges (3), (4) would be expected to give rise to deformations and to electric charges of the piezoelectric cylinder corresponding to a displacement $\mathbf{u}$ and to an electric potential field φ which are independent on x_3 and have the the form

$$\mathbf{u} = (0,0,\ u) \qquad with \qquad u = u(x_1,\ x_2,t):\ \Omega\times[0,T]\to \mathrm{IR}, \quad (5)$$

$$\varphi = \varphi(x_1,\ x_2,t)\ :\ \Omega\times[0,T]\to \mathrm{IR}. \quad (6)$$

Such kind of deformation, associated to a displacement field of the form (3), is called an antiplane shear.

The infinitesimal strain tensor is denoted $\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))$ and the stress field by $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_{ij})$. We also denote by $\mathbf{E}(\varphi) = (E_i(\varphi))$ the electric field and by $\mathbf{D} = (D_i)$ the electric displacement field. Here and below, in order to simplify the notation, we do not indicate the dependence of various functions on x_1, x_2, x_3 or t and we recall that

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}),\ \ E_i(\varphi) = -\varphi_{,i}.$$

The material's is modeled by the following electro-elastic constitutive law with Tresca friction law

$$\sigma = \lambda\,(\mathbf{tr} \in(\mathbf{u}))\mathbf{I} + 2\mu \in(\mathbf{u}) - \mathbf{E}^*\mathbf{E}(\varphi), \quad (7)$$

$$\mathbf{D} = \mathbf{E}\boldsymbol{\varepsilon}(\mathbf{u}) + \beta\mathbf{E}(\varphi), \quad (8)$$

where λ and μ are the Lame coefficients $\varepsilon(\mathbf{u}) = (\in ij\,(\mathbf{u}))$, $\mathbf{I}$ is the unit tensor in IR^3, β is the electric permittivity constant, $\mathbf{E}$ represents the third-order piezoelectric tensor and $\mathbf{E}^*$ is its transpose. In the antiplane context (5), (6), using the constitutive equations (7), (8) it follows that the stress field and the electric displacement field are given by

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & \boldsymbol{\sigma}_{13} \\ 0 & 0 & \boldsymbol{\sigma}_{23} \\ \boldsymbol{\sigma}_{31} & \boldsymbol{\sigma}_{32} & 0 \end{pmatrix}, \quad (9)$$

$$\mathbf{D} = \begin{pmatrix} eu,_1 - \beta\varphi,_1 \\ eu,_2 - \beta\varphi,_2 \\ 0 \end{pmatrix} \quad (10)$$

where

$$\boldsymbol{\sigma}_{13} = \boldsymbol{\sigma}_{31} = \mu\partial_{x_1}u$$

and

$$\boldsymbol{\sigma}_{23} = \boldsymbol{\sigma}_{32} = \mu\partial_{x_2}u.$$

We assume that

$$\mathbf{E}\boldsymbol{\varepsilon} = \begin{pmatrix} e(\varepsilon_{13}+\varepsilon_{31}) \\ e(\varepsilon_{23}+\varepsilon_{32}) \\ e\varepsilon_{33} \end{pmatrix}\ \forall\boldsymbol{\varepsilon} = (\varepsilon_{ij}) \in \mathbf{S}^3, \quad (11)$$

where e is a piezoelectric coefficient. We also assume that the coefficients μ, β and e depend on the spatial variables x_1, x_2, but are independent on the spatial variable x_3. Since $\mathbf{E}\boldsymbol{\varepsilon}\cdot\mathbf{v} = \boldsymbol{\varepsilon}\cdot\mathbf{E}^*\mathbf{v}$ for all $\boldsymbol{\varepsilon}\in\mathbf{S}^3$, $\mathbf{v}\in IR^3$, it follows from (e) that

$$\mathbf{E}^*\mathbf{v} = \begin{pmatrix} 0 & 0 & ev_1 \\ 0 & 0 & ev_2 \\ ev_1 & ev_2 & ev_3 \end{pmatrix}\ \forall\mathbf{v} = (v_i) \in \mathrm{IR}^3. \quad (12)$$

We assume that the process is mechanically quasistatic and electrically static and therefore is governed by the equilibrium equations

$$\mathbf{Div}\sigma + \mathbf{f}_0 = 0,\ D_{i,i} - q_0 = 0 \ \text{ in } \ B\times[0,T]\ ,$$

where $\mathbf{Div}\sigma = (\sigma_{ij,j})$ represents the divergence of the tensor field σ. Taking into account (1), (3), (5), (6), (9) and (10), the equilibrium equations above reduce to the following scalar equations

$$\mathbf{div}(\mu\nabla u + e\nabla\varphi) + f_0 = 0,\ in\ \Omega\times[0,T], \quad (13)$$

$$\mathbf{div}(e\nabla u - \beta\nabla\varphi) = q_0,\ in\ \Omega\times[0,T]. \quad (14)$$

Here and below we use the notation

$\mathbf{div}\tau = \tau_{1,1} + \tau_{1,2}$ in $\tau = (\tau_1(x_1, x_2), \tau_2(x_1, x_2))$

and

$\nabla v = (v_1, v_2)$, $\partial_{t/} v = v_1 \nu_1 + v_2 \nu_2$ for $v = v(x_1, x_2)$

We now describe the boundary conditions. During the process the cylinder is clamped on $\Gamma_1 \times (-\infty, +\infty)$ and the electric potential vanish on $\Gamma_1 \times (-\infty, +\infty)$; thus, (5) and (6) imply that

$$u = 0 \text{ on } \Gamma_1 \times [0,T], \quad (15)$$

$$\varphi = 0 \text{ on } \Gamma_a \times [0,T]. \quad (16)$$

Let $\boldsymbol{\nu}$ denote the unit normal on $\Gamma \times (-\infty, +\infty)$. We have

$$\nu = (\nu_1, \nu_2, 0) \textit{ with } \nu_i = \nu_i(x_1, x_2) : \Gamma \to \mathrm{IR},\ i = 1,2. \quad (17)$$

For a vector $\mathbf{v}$ we denote by v_ν and $\mathbf{v}_\tau$ its normal and tangential components on the boundary, given by

$$v_\nu = \mathbf{v} \cdot \boldsymbol{\nu}, \ \mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}. \quad (18)$$

For a given stress field $\boldsymbol{\sigma}$ we denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and the tangential components on the boundary, that is

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \ \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}. \quad (19)$$

From (9), (10) and (17) we deduce that the Cauchy stress vector and the normal component of the electric displacement field are given by

$$\boldsymbol{\sigma}\boldsymbol{\nu} = (0, 0, \mu\partial_\nu u + e\partial_\nu \varphi), \ \mathbf{D} \cdot \boldsymbol{\nu} = e\partial_\nu u - \beta\partial_\nu \varphi. \quad (20)$$

Taking into account (2), (4) and (20), the traction condition on $\Gamma_2 \times (-\infty, \infty)$ and the electric conditions conditions on $\Gamma_b \times (-\infty, \infty)$ are given by

$$\mu\partial_\nu u + e\partial_\nu \varphi = f_2 \text{ on } \Gamma_2 \times 0, T],[(21)$$

$$e\partial_\nu u - \beta\partial_\nu \varphi = q_2 \text{ on } \Gamma_b \times [0,T).(22)$$

We now describe the frictional contact condition and the electric conditions on $\Gamma_3 \times (-\infty, +\infty)$. First, from (5) and (17) we infer that the normal displacement vanishes, $u_\nu = 0$, which shows that the contact is bilateral, that is, the contact is kept during all the process. Using now (5) and (17)-(19) we conclude that

$$\mathbf{u}_\tau = (0,0,u), \ \boldsymbol{\sigma}_\tau = (0,0,\sigma_\tau) \quad (23)$$

where

$$\sigma_\tau = (0, 0, \mu\partial_\nu u + e\partial_\nu \varphi).$$

We assume that the friction is invariant with respect to the x_3 axis and is modeled with Tresca's friction law, that is

$$\boldsymbol{\sigma}_\tau(t) = \begin{cases} 0, \text{if u} = 0, \\ -g|u|^{s-1}, & \textit{if} \ \ \text{u} \neq 0 \ \text{ on } \Gamma_3 \times (0,T).(24) \end{cases}$$

Here $g : \Gamma_3 \to \mathrm{IR}_+$ is a given function, the friction bound, and $\mathbf{u}_\tau$ represents the tangential velocity on the contact boundary. Using now (23) it is straightforward to see that the friction law (24) implies

$$\mu\partial_\nu u + e\partial_\nu \varphi = \begin{cases} 0, \text{if u} = 0, \\ -g|u|^{s-1}, & \textit{if} \ \ \text{u} \neq 0 \ \text{ on } \Gamma_3 \times (0,T).(25) \end{cases}$$

Next, since the foundation is electrically conductive and the contact is bilateral, we assume that the normal component of the electric displacement field or the free charge is proportional to the difference between the potential on the foundation and the body's surface. Thus,

$$\mathbf{D} \cdot \boldsymbol{\nu} = q_2 \text{ on } \Gamma_b \times (0,T),$$

Then, we get

$$\begin{pmatrix} eu_{,1} - \beta\varphi_{,1} \\ eu_{,2} - \beta\varphi_{,2} \\ 0 \end{pmatrix} . \boldsymbol{\nu} = q_2 \text{ on } \Gamma_b \times (0,T). \ (26)$$

Finally, we use (20) and the previous equality to obtain

$$e\partial_\nu u - \beta\partial_\nu \varphi = q_2 \text{ on } \Gamma_b \times (0,T). \quad (27)$$

We collect the above equations and conditions to obtain the following mathematical model which describes the antiplane shear of an electro-viscoelastic cylinder in frictional contact with a conductive foundation.

Problem P. Find the displacement field $u : \Omega \to \mathrm{IR}$ and the electric potential $\varphi : \Omega \to \mathrm{IR}$ such that

$$\mathbf{div}(\mu \nabla u) + \mathbf{div}(e \nabla \varphi) + f_0 = 0, \ \textit{in}\ \Omega, \ (28)$$

$$\mathbf{div}(e \nabla u) - \mathbf{div}(\alpha \nabla \varphi) = q_0 \ \mathbf{in}\ \Omega, \ (29)$$

$$u = 0 \text{ on } \Gamma_1, (30)$$

$$\mu\partial_\nu u + e\partial_\nu \varphi = f_2 \text{ on } \Gamma_2, (31)$$

$$\mu\partial_\nu u + e\partial_\nu \varphi = \begin{cases} 0, \text{if u} = 0, \\ -g|u|^{s-1}, & \textit{if} \ \ \text{u} \neq 0 \ \text{ on } \Gamma_3, (25) \end{cases}$$

$$\varphi = 0 \text{ on } \Gamma_a, \ (33)$$

$$e\partial_\nu u - \alpha\partial_\nu \varphi = q_2 \quad \text{on } \Gamma_b. \quad (34)$$

Note that once the displacement field u and the electric potential φ which solve Problem P are known, then the stress tensor σ and the electric displacement field $\mathbf{D}$ can be obtained by using the constitutive laws (9) and (10), respectively.

3. VARIATIONAL FORMULATION

For a real Banach space $(X,\|.\|_X)$ we use the usual notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$ where $1 \le p \le \infty,\ k = 1,2,...$; we also denote by $C([0,T];X)$ the space of continuous and continuously differentiable functions on $[0,T]$ with values in X, with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X$$

and we use the standard notations for the Lebesgue space $L^2(0,T;X)$ as well as the Sobolev space $W^{1,2}(0,T;X)$. In particular, recall that the norm on the space $L^2(0,T;X)$ is given by the formula

$$\|u\|^2_{L^2(0,T;X)} = \int_0^T \|u(t)\|_X^2\, dt$$

and the norm on the space $W^2(0,T;X)$ is given by the formula

$$\|u\|^2_{W^{1,2}(0,T;X)} = \int_0^T \|u(t)\|_X^2\, dt + \int_0^T \|\dot u(t)\|_X^2\, dt. \quad (38)$$

Finally, we suppose the argument X when $X = IR$; thus, for example, we use the notation $W^2(0,T)$ for the space $W^2(0,T;IR)$ and the notation $\|.\|_{W^2(0,T)}$ for the norm $\|.\|_{W^2(0,T;\mathrm{R})}$.

In the study of the **Problem P** we assume that the viscosity coefficient satisfy:

and the electric permittivity coefficient satisfy

$$\beta \in L^\infty(\Omega) \text{ and there exists } \beta^* > 0 \text{ such that } \beta(\mathbf{x}) \ge \beta^* \text{ a.e.} \mathbf{x} \in \Omega.$$
(39)

We also assume that the Lame coefficient and the piezoelectric coefficient satisfy

$$\mu \in L^\infty(\Omega) \quad (40)$$
and
$$\mu(\mathbf{x}) > 0 \ \text{ a.e.} \mathbf{x} \in \Omega, (41)$$
$$e \in L^\infty(\Omega). (42)$$

The forces, tractions, volume and surface free charge densities have the regularity

$$f_0 \in L^2(\Omega), (43)$$
$$f_2 \in L^2(\Gamma_2), (44)$$
$$q_0 \in L^2(\Omega), (45)$$
$$q_2 \in L^2(\Gamma_b). (46)$$

The friction bound function g satisfies the following properties

$$g \in L^\infty(\Gamma_3) \text{ and } g(\mathbf{x}) \ge 0 \text{ a.e.} \mathbf{x} \in \Gamma_3. (47)$$

and, moreover,

$$a_\mu(u_0, v)_V + j(v) \ge (f(0), v)_V \quad \forall v \in V. (48)$$

We define now the functional $j : V \to \mathrm{IR}_+$ given by the formula

$$j(v) = \frac{1}{s+1}\int_{\Gamma_3} g\,|v|^{s+1} da \quad \forall v \in V. (49)$$

We also define the mappings $f \in V$ and $q \in W$, respectively, by

$$(f, v)_V = \int_\Omega f_0 v\, dx + \int_{\Gamma_2} f_2 v\, da, (50)$$

and

$$(q,\psi)_W = \int_\Omega q_0 \psi\, dx - \int_{\Gamma_b} q_2 \psi\, da, (51)$$

for all $v \in V$, $\psi \in W$ and $t \in [0,T]$. The definition of f and q are based on Riesz's representation theorem; moreover, it follows from assumptions by (42)-(43), that the integrals above are well-defined and

$$f \in L^2(\Omega), (52)$$
$$q \in L^2(\Omega). (53)$$

Next, we define the bilinear forms a_μ : $V \times V \to \mathrm{IR}$, a_e : $V \times W \to \mathrm{IR}$, , and a_α : $W \times W \to \mathrm{IR}$, by equalities

$$a_\mu(u,v) = \int_\Omega \mu \nabla u \cdot \nabla v\, dx, (54)$$
$$a_e(u,\varphi) = \int_\Omega e \nabla u \cdot \nabla \varphi\, dx, (55)$$
$$a_\alpha(\varphi,\psi) = \int_\Omega \beta \nabla \varphi \cdot \nabla \psi\, dx, (56)$$

for all $u, v \in V$, $\varphi,\psi \in W$. Assumptions (49)-(51) imply that the integrals above are well defined and, using (37) and (18), it follows that the forms a_μ and a_e are continuous; moreover, the forms a_μ and a_α are symmetric and, in addition, the form a_α is W -elliptic, since

$$a_\alpha(\psi,\psi) \ge \alpha^* \|\psi\|_W^2 \quad \forall \psi \in W. \quad (57)$$

4. MAIN RESULTS

The variational formulation of **Problem P** is based on the

Lemma 1 For all (u,φ) in space $X = V \times W$, then we get

$$\int_\Omega \mu\nabla u.\nabla(v-u)dx + \int_\Omega e\nabla\varphi.\nabla(v-u)dx + \frac{1}{s+1}\int_{\Gamma_3}(|v|^{s+1} - |u|^{s+1})da \geq$$
$$\int_\Omega f_0(v-u)dx + \int_{\Gamma_2} f_2(v-u)da, \forall(v,\psi) \in X = V \times W, \forall(u,\varphi) \in X = V \times W.(58)$$

Proof. We introduce relation (50) in the previous relation, then, we have

$$a_\mu(u,v-u) + a_e(\varphi,v-u) + j(v) - j(u) \geq (f,v-u)_V, \forall v \in V, \forall\varphi \in W,(71)$$

which conclude the proof of lemma 1.

Lemma 2. For all element $\psi \in W$ and for all $(u,\varphi) \in V \times W$, then, we have

$$a_e(\varphi,\psi) - a_\beta(u,\psi) = (q,\psi)_W, \forall\psi \in W.(72)$$

Proof. It is immediately by using (29), (33) and (34).

We collect the above equations and conditions to obtain the following variational formulation which describes the antiplane shear of an electro-viscoelastic cylinder in frictional contact with a conductive foundation.

Problem PV 1**.** Find a displacement field $u : \Omega \to V$ and an electric potential field $\varphi : \Omega \to W$ such that

$$a_\mu(u,v-u) + a_e(\varphi,v-u) + j(v) - j(u) \geq (f,v-u)_V, \forall v \in V, \forall\varphi \in W,(77)$$

$$a_e(\varphi,\psi) - a_\beta(u,\psi) = (q,\psi)_W, \forall\psi \in W.(78)$$

Let now using the bilinear form:

$$a(.,.): X \times X \to R$$
$$(x,y) \mapsto a(x,y) = a_\mu(u,v) + a_e(\varphi,v) + a_\beta(\varphi,\psi) - a_e(u,\psi), \forall x = (u,\varphi) \in X, \forall y = (v,\psi) \in X,(79)$$

the functional

$$J(.): X \to R$$
$$x \mapsto J(x) = j(u), \forall x = (u,\varphi) \in X,(80)$$

and the function

$$F = (f,q) \in X.(81)$$

Now, using notations (79)-(81), the Problem (77)-(78) take the final form:

Problem PV 2**.** Find a couple $x = (u,\varphi) \in X$ such that

$$a\ (x,y-x) + J(x) - J(y) \geq (F,y-x)_X, \forall y \in X.(82)$$

Theorem 3. The **Problem** PV 1 and **Problem** PV 2 are equivalent.

Proof. We have two step to proof our Theorem.

Step 1: Problem PV 1 $\Rightarrow$ **Problem** PV 2

In the first step we ill suppose that $x = (u,\varphi) \in X$ is solution of **Problem** PV 1**.** We change in (78) the element $\psi \in W$ by $(\psi - \varphi) \in W$ and we add the resulting equation to the two sides of the inequality (77), hence, we obtain:

$$a_\mu(u,v-u) + a_e(\varphi,v-u) + a_\beta(\varphi,\psi-\varphi) - a_\beta(u,\psi-\varphi) +$$
$$+ j(v) - j(u) \geq (f,v-u)_V + (q,\psi-\varphi)_W, \forall v \in V, \forall\varphi \in W.(83)$$

Using now notations (79), (80) and (81) then for all $\psi \in W$ and for all $y \in X$, we get

$$a\ (x,y-x) + J(x) - J(y) \geq (F,y-x)_X, \forall y \in X.(84)$$

which conclude the proof of the first step.

Step 2: Problem PV 2 $\Rightarrow$ **Problem** PV 1

In ths second step we will suppose that $x = (u,\varphi) \in X$ is solution of **Problem** PV 2**.** We change the bilinear form $a(.,.)$ by (79), $(F,y-x)_X$ by (81) and the functional $J(.)$ by (80); then, for all $(v,\psi) \in X$, we obtain

$$a_\mu(u,v-u) + a_e(\varphi,v-u) + a_\beta(\varphi,\psi-\varphi) +$$
$$+ j(v) - j(u) \geq (f,v-u)_V + (q,\psi-\varphi)_W, \forall v \in V, \forall\varphi \in W.(85)$$

We test in the last inequality (85) with $\psi = \varphi$, then we obtain (77). Next, we take $v = u$ and $\psi - \varphi = \varphi \pm \psi - \varphi$ in (84), it follows that for all $\psi \in W$:

$$a_\beta(\varphi,\pm\psi) - a_e(\varphi,\pm\psi) \geq (q,\pm\psi)_W, \forall v \in V, \forall\varphi \in W,(86)$$

which conclude the proof of the second. Then, the **Problem** PV 1 and **Problem** PV 2 are equivalent.

Our main existence and uniqueness result, which we state now and prove in the next section, is the following:

Theorem 4. Assume that (39)-(57) hold. Then the variational **Problem** PV 2 possesses a unique solution $x = (u,\varphi) \in X$ satisfies

$$a\ (x,y-x) + J(x) - J(y) \geq (F,y-x)_X, \forall y \in X.(87)$$

We note that an element $x = (u,\varphi)$ which solves Problem PV 1 is scalled a weak solution of the antiplane contact Problem PV 1. We conclude by Theorem 3 that the element $x = (u,\varphi)$ also solves Problem PV 2, then the element x is called a weak solution of the antiplane contact

Problem $PV2$. Hence, the antiplane contact Problem P has a unique weak solution, provided that (39)-(57).

Proof of Theorem 4.

The Proof of Theorem 4 which will be carried out in several steps and it is immediately to obtain our result of existence and uniqueness of the weak solution.

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Abu Nasr Mansur b. 'Ali b. ‘Iraq (lived circa 950-1036) and Abu l-Rayhan al-Biruni (lived from 973-after 1050) as students, teachers, and companions

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Abstract

This paper discusses issues of a history of teaching the mathematical sciences in the so-called classical or foundational period of Islamicate societies. This history does not exist so far since all research on the history of the mathematical sciences in Islamicate societies before 1300 during the last fifty years focused on translations from Greek and occasionally other languages into Arabic as well as innovative methods, theories and instruments. I discuss how to build such a history on the basis of a case study.

Key words: *history of teaching; mathematical sciences; Islamicate societies; foundational period; Abu Nasr b. 'Ali b. 'Iraq; Abu l-Rayhan al-Biruni; methodology*

The history of teaching the mathematical sciences in Islamicate societies is still in its infancy. Most academic research concentrates on the editions, translation, and interpretation of texts, rarely studying at any depth they social, cultural, or economic conctexts of the authors and their contemporaries. So far, my own work focused primarily on the mathematical sciences at madrasas in Egypt and Syria after 1200, since their presence in classes taught there was denied by earlier researchers and since most of the biographical material related to the mathematical sciences at madrasas has been written by scholars from those regions.

In this paper, I focus on an earlier period which as a rule is exlusively approached from the perspective of high-level mathematical achievements, i.e. research. Although we know the one or the other detail with regard to teaching, there is paper dedicated to this part of mathematical practice in any Islamicate society before 1200. Working on an introductory book on the history of teaching the sciences in Islamicate societies until 1700, I am not able to fully compensate this lack of research for such a long period of time. Hence, I decided to treat a few selected examples. One pair of well-known scholars, whose engagement in teaching I wish to include in that book, are the friends and colleagues Abu Nasr Mansur b. 'Ali b. 'Iraq and Abu l-Rayhan al-Biruni. I present in this paper my survey on their activities in this regard with some reflections on methodological issues.

The methodological issues that I address concern the biographies of the two men, how to differentiat teaching from research texts, and problems of how to understand relationships and forms of style and rhetoric.

1. Biographical issues

Our knowledge about the ancestors of Abu Nasr Mansur b. 'Iraq (from now on: Abu Nasr or Ibn ‘Iraq) and therewith his family rests on two different kinds of sources: al-Biruni's works and archeological excavations with its recovered objects and sites. These two sources of knowledge do not agree in most of their details, but it is not easy to determine which problem causes the greater inaccuracies: al-Biruni's lack of ancient sources and his prejudices against the Arabic invaders of the eighth and ninth centuries or the loss of most written sources of pre-Islamic Khwarazm. (Bosworth 2011, vol. 1, fasc. 7, 743-745) Hence, for the purpose of this paper I simply register that Abu Nasr possibly was a prince of the ruling dynasty of the Khwarazmshahs (Afrighid family) and that al-Biruni is believed to have been born in a suburb of Kat, the ancient capital of this state. Al-Biruni describes himself in a poem as an orphan and his mother as coming from a lowly family.

Ibn ‘Iraq's family was overthrown in 995 by the neighboring Ma'munid emir of Gurganj, possible as ripple effect of the westward drive of Nomadic tribes in central Asia. For Biruni began a long time of traveling. He apparently went first to Rayy, which was a Buyyid stronghold. (Bulgakov 1966, 13) But in 997, he was back in Kat observing in cooperation with Abu l-Wafa' (948-998) in Baghdad a lunar eclipse. One year later, he left again moving to the court of the Ziyarid ruler Qabus b. Vushmgir (reigned 977-981, 997-1012) in Gurgan at the south-eastern shores of the Caspian Sea. Although the Ziyarid court was a flourishing cultural and scholarly center in that time, al-Biruni was not too happy there complaining in his later works about the lack of instruments and opportunities for scholarly research. (Rozenfel'd, Rozhanskaya and Sokolovskaya 1971, 11) Possibly receiving an invitation, al-Biruni returned during the first decade of the eleventh century (the precise year is contested) to Khwarazm serving until 1017 at the last Ma'munid ruler's court in Urgench as a boon companion, princely adviser, and skilled diplomat. (Bosworth 2011, vol. 4, 274-276) There he made the acquaintance of Ibn Sina and one of his teachers, Abu Sahl 'Isa b. Yahya al-Masihi (died 1012). At some unknown point in time, Ibn ‘Iraq came to serve at the court in Kat. There he survived the overthrow of his family and became a courtier of the Ma'munid rulers 'Ali b. Ma'mun (reigned 997-1009) and Abu l-'Abbas Ma'mun (reigned 1009-1017).

In 1017, the political turmoil of those years once again changed the lives of both men dramatically. Since 1008, the new Ghaznavid dynasty with their capital in Ghazna (today in Afghanistan) was the new regional superpower. The Khwarazmshahs tried to maintain their dominion by trying to forge alliances both with the Abbasid caliphs in Baghdad and the Ghaznavid ruler Mahmud (reigned 997-1030). This policy finally failed in 1017 when Mahmud sent an ultimatum to Abu al-'Abbas Ma'mun demanding to be accepted as Ma'mun's overlord, to receive a hefty tribute, and to be sent the group of leading scholars assembled at Ma'mun's court. According to an anecdote told by Nizami Arudi (flourished between 1110 and 1161) from Samarqand, Ma'mun read the missive to the said scholars and offered them the opportunity to make their own decision. (Browne 1921, 85-87) Ibn Sina and Abu Sahl al-Masihi decided to flee in southwestern direction. During this flight, Abu Sahl died due to thirst and exhaustion. Al-Biruni, Ibn ‘Iraq, and at least two further scholars chose to go to Mahmud's court in Ghazna. This report is ostensibly wrong, since IbnSina had left al-Ma'mun's court in 1012, and Abu Sahl al-Masihi had died during that voyage towards the Ziyarid court in Gurganj. Moreover, al-Ma'mun paid his willingness to obey Mahmud with his life. His nobles and army leaders rebelled. He was killed in March 1017, and his young nephew Abu l-Harith Muhammad b. 'Ali (died 1017?) was enthroned. Four months later, Mahmud's army conquered Khwarazm "avenging his brother-in-law's murder". (Bosworth 2011, vol. 1, fasc. 7, 744) It seems that only then al-Biruni, Ibn ‘Iraq, and two other scholars were moved to Ghazna, where they spent the remaining decades of their lives with varying fortunes in the Ghaznavid empire and at its dynasty's court.

2. Teachers, Students, Companions

On the basis of Abu Nasr's and al-Biruni's later scholarly papers we can gain some glimpses of the kind of education both boys received. It included classical Arabic, New Persian, and the mathematical sciences. Since by then the Islamic faith was widely spread in Khwarazm and the ruling dynasty had converted to it a century earlier, they will also have learned to recite the Qur'an and studied other religious literature. Poetry also seems to have been part of their education. Philosophy and medicine, on the other hand, do not figure prominently in their work and thus may not have been an important part of their school years. In addition, al-Biruni remembers as an old man, being a curious child he went to visit someone from Byzantium who happened to live in Kat, bringing plants, seeds, and other items to him, asking for their names in his mother tongue, and writing them down. (Bulgakov 1972, 31)

Given this scarcity of information, it is very difficult to determine who their teachers were. Ibn ‘Iraq allegeldy received his mathematical and astronomical training from one of the leading scholars of the mathematical sciences of the Buyid era during the tenth century - Abu l-Wafa' (940-998) from Buzjan in eastern Iran. Since Abu l-Wafa' moved in 959 to Baghdad, when Ibn ‘Iraq was at best nine years old, this is difficult to accept, because there is no evidence that Ibn ‘Iraq spent a longer period of his life in the Abbasid capital. Thus, if the two men met and talked about the mathematical sciences, this must have happened somewhere in Iran, possibly at one of the Buyid courts. At an unidentified date, but before 998, Abu Nasr wrote a treatise on azimuths about which he said that Abu l-Wafa' had read it. (Samsò 1969, 28-29)

In 997, one year before Abu l-Wafa's death, al-Biruni exchanged letters with him about a joint astronomical project. During this decade, al-Biruni also constructed astronomical instruments and perhaps a terrestrial globe and observed the altitude of the Sun in his hometown for the Spring and Autumn equinoxes. (Bulgakov 1972, 27) Thus, al-Biruni's higher education in the mathematical sciences must have taken place before 990, when he celebrated his seventeenth birthday, or at least must have had been in its advanced stage by then. Perhaps his early observations, constructions of instruments and a globe as well as his early texts written in the same period may be regarded as components of his training as a young scholar.

The following quote describes one of these early observations made by al-Biruni in the age of 22 according to his recital in the *Geodesy* situating it in the period of the overthrow of Ibn ‘Iraq's family:

> I measured the solar altitude twice. In the first time, I measured it in the village called Bushkanz (?) at the western shores of the Jayhun [Amu Darya], between Jurjaniyya and the [main] city of Khwarazm [i.e. Kat] in the year three hundred eighty four or three Hijra, with a circle of fifteen cubits in the plane of the horizon. I measured on the shortest shadow the Sun cast its greatest altitude and found it to be 71°59'45". Then at the same day, I got the value of the shadow, when it reached the line of day and night equality. But I forgot it because of the riots and commotion [in Khwarazm], which forced me to leave the country and to interrupt the works. But I remember that the value of the greatest inclination, which I got from these two [altitudes], was 23°35'45" and the longitude of the said village was 41°36'. (Bulgakov 1972, 32)

Analyzing several of Ibn ‘Iraq's and al-Biruni's extant writings, Julio Samsò pointed to al-Biruni's unique approach to several scholars in his environment. They allegedly wrote treatises in al-Biruni's name. (Samsò 1969, 18-19) Two of them are said to have written each twelve such shorter or longer treatises: Ibn ‘Iraq and Abu Sahl al-Masihi. Al-Biruni asked Abu Nasr for an explanation and proof of problems in works by earlier authors or in practices of craftsmen or for determining their errors. Ibn ‘Iraq appears here in a double role - that of a teacher and perhaps the technically more apt scholar and that of an adjunct of al-Biruni whom - Samsò believes - Ibn ‘Iraq wished to free from detailed, pedantic labor for more ambitious projects. (Samsò 1969, 21) If Samsò's reflections on the possible dating of some of these texts are correct then al-Biruni continued to ask for such services over a period of almost three decades, i.e. when he was already almost 50 years old and Ibn ‘Iraq complied with such requests even when he was beyond the seventies.

It is impossible to decide whether there was a shift from one relationship to the other or whether all those texts reflect Ibn Ira's auxiliary work for al-Biruni, because except for certain passages where Abu Nasr addresses al-Biruni directly, his language otherwise is very formal. It agrees with that of Euclid's *Elements* by presenting a proposition, followed sometimes by an explanation, and then continued by a proof with the standard terminating formula that this was what one wished to do, explain or know. The direct talk to al-Biruni clarifies that Ibn ‘Iraq responds to questions and requests posed to him by al-Biruni. Their style corresponds to the question-and-answer texts written in Baghdad since the ninth century for students and curious acquaintances. In his *Epistle about the Intersections of the Azimuthal Circles on the Astrolabe*, Ibn ‘Iraq writes for instance:

> You said, may God honor you, that the procedures based on the calculation for finding on the astrolabe the intersection of the azimuthal circles with the horizon and the tropics of Capricorn as well as the procedures of the craftsmen for obtaining this were arrived at according to your knowledge without proofs, which (would) make you trust them. (You added) that even when they were related to the most reputed people in the profession, you did not achieve (complete) certainty that they were free of errors and (mistakes due to) the lack of attentention by copyists of which the copies are only rarely free, unless you would obtain proofs and (take the time to) consider the test of such rules.
>
> You asked me to explain to you that what I clearly see in this matter and I acceded to your request ... (Samsò 1969, 20)

Such direct approaches to al-Biruni are also found occasionally between theorems or proofs. Al-Biruni asked Ibn ‘Iraq in most cases for either one of the two points raised in the just quoted introduction: to prove that what an earlier scholar had provided was correct or to correct the errors and ommissions found in a copy of their work available to al-Biruni. (Samsò 1969, 28-32) When answering al-Biruni's questions or dealing with the tasks he posed, Ibn ‘Iraq excerpted earlier research texts, commented on them, corrected them, or added proofs to them, when they had not been provided, and proposed his own solutions. He clearly treated the matter in a thorough manner, merging thus the genre of questions and answers with that of a textbook.

3. Al-Biruni's questions to Ibn Sina

Of a different nature are the three letters with philosophical, astronomical, optical, and related scientific questions which al-Biruni sent to Ibn Sina. They incorporate curiosity, challenge, dissatisfaction, and criticism. It is believed that this exachange took place around 1000, when al-Biruni was at the court of Qabus b. Vushmgir and Ibn Sina, about 20 years old, was still in Bukhara. (Glick 2005, 88; for the date of Ibn Sina's flight from Bukhara see Gutas 1987-1988, 334) Ibn Sina is said to have defended "orthodox Aristotelian" views, while al-Biruni showed his "independent" mind by accusing Aristotle to rely too much on authority and to abstain from making observations. (Glick 2005, 88) The year 1000 as a date for the exchange is, however, doubtful in terms of chronology. While Ibn Sina answered the first two letters with eighteen questions in person, he gave al-Biruni's last reply to his student Abu Sa'id Ahmad b. 'Ali al-Mas'umi (late tenth-first half eleventh centuries), exasperated and angry about al-Biruni's choice of words and continued challenge. (Reisman 2007, 197) The fact that al-Mas'umi answered al-Biruni's third letter also contradicts an early dating of the exchange, since Ibn Sina seems to have had students whose names we know only from 1013 onwards. (Gutas 2014, 19)

With regard to Ibn Sina's and al-Biruni's intellectual positions in these letters, things are neither simple and clear cut. One strong current in them is al-Biruni's rejection of any theoretical claim that can be shown to deviate from empirical observations. Ibn Sina, on the other hand, points to al-Biruni's weaknesses in his knowledge and interpretation of philosophical theories and tries to determine the books from which he might have derived his views. This does not mean that Ibn Sina does not mention experiments in his replies. But he embeds them as a rule in references to mostly Aristotelian books, in particular *On the Heavens*, *On the Soul*, *On Generation and Corruption*, *Meteorology*, and *On Sense and Sensibility*.

The overall character of al-Biruni's questions and Ibn Sina's and al-Ma'umi's answers is that of a scientific dispute, in which al-Biruni mostly, but not always challenges Aristotelian positions. But he also asked questions free of polemical character, which might imply that one reason for this exchange might indeed have been to acquire knowledge which he did not possess, i.e. to learn. The following is an example for such a non-polemical question:

> The Tenth Question: What causes transformation of elements into each other? Is it the result of their proximity or intermingling or some other process? Let us take the example of air and water: when water transforms into air, does it become air in reality, or is it because its particles spread out until they become invisible to the sight so that one cannot see these separate particles? (https://internationsocietyofclassicalastrologers.files.wordpress.com/2013/04/al-biruni_ibn-sina-correspondence.pdf, unpaginated, but numbered according to entries: 42)

Ibn Sina's reply here begins with a brief summary of his views, then names Aristotle's books (*On Generation and Corruption*; *Meteorology*; *On the Heavens*, Book III) where al-Biruni could find more detailed information, and finally offers an example with the aim to clarify the philosophical methods and demonstrations used for this problem. In this sense, we can consider the letters between al-Biruni and Ibn Sina as documents of high level teaching and learning perhaps in a postdoctoral phase, to use our own concepts.

Ibn Sina's answer:

> The transformation of elements into one another does not occur the way you mentioned. Water does not transform into air by the separation and the spread of its particles in the air until they disappear from the sight; rather, the water particles take off their watery identity and put on an airy identity. For more details, one can see the commentaries on *Kitab al-Kawn wa'l-Fasad* and *Kitab al-Athar al-'ulwiyah* and the Book III of *Kitab al-Sama'*. But here I clarify this case according to their methods and the following logical example that they used to prove their sayings.
>
> Increase in the size of bodies <can be explained> by means of an example: <Suppose>, we took a flask filled with water, sealed it tightly and exposed it to intense heat. The water particles in the flask would expand and crack the flask because their size increased when they transformed into air. This happened either because of the spread of the space between the water particles, but not because of the spread of particles. But the void is impossible; therefore, it is necessary that the latter is true. <Thus> the reason for transformation <of water into air> is not the spread of its particles, but the acceptance of another identity by its parts.
>
> If it would be said that air or something else entered the flask and increased its volume, we would say: that is impossible because a full container cannot accept another body inside it until it is emptied of the first occupant, and the water cannot leave the flask because it is tightly sealed and there is no way out. I observed a little flask. We tightly sealed it and put it in a kiln. It did not take long before it cracked and everything that was in it exploded into the fire. And it is known that nothing mixed with the particles of the water that were inside the flask that could cause a change, because, firstly, the fire was not inside the flask and, secondly, it did not enter it because there was no way into the flask. It is, therefore, obvious that this transformation occurred through a change in the air and fire natures of <air and fire> and not through the spread of parts. I have provided an example which supports Aristotle's views on the generation and change as parts of nature; and this suffices, for further elaboration would demand tenuous efforts. Many objections could arise in this matter and if you encounter any, please convey your questions and I would explain to you, God willing.

(https://internationsocietyofclassicalastrologers.files.wordpress.com/2013/04/al-biruni_ibn-sina-correspondence.pdf, entry 43-45)[1]

4. Science for Rayhana

Between 1027 and 1029, al-Biruni wrote in Ghazna a voluminous book in form of 530 questions and answers on astronomy and astrology calling it *The Book on the Understanding of the Principles of the Art of the Stars* [from now on: *The Book on the Stars*]. The recipient was a young woman by name of Rayhana, daughter of al-Hasan. According to al-Biruni's introduction to the book, Rayhana was from Khwarazm and had asked for instruction. Al-Biruni must have highly appreciated her intelligence and capacity to learn since instead of writing a little epistle for her, he wrote a full-fleged course about the four sciences of the quadrivium extended in various directions. He introduced Rayhana into plane and solid geometry, geometry of the sphere, theory of proportions, number theory, systems of counting and calculating, algebra, Ptolemaic astronomy, timekeeping, and the astrological doctrines of Hellenistic and late antique, Indian, Iranian, and Muslim authors. Some questions also pertain to balances and weighing, roots and powers defined in accordance with definitions of Book X of Euclid's *Elements*, or arithmetical rules not found in Greek treatises. Al-Biruni justifies his choice of the dialogical format as being better suited for learning and easier to understand. (Abu Raykhan Beruni 1975, 21) Given the broad scale and the complexity of knowledge that al-Biruni presented to the young woman, this was certainly a good choice.

Since it is impossible to survey the entire 530 questions and answers here, a few need to suffice for presenting al-Biruni as a teacher. I chose the briefest ones, ignoring those that run over a page or more. When reading different answers, it becomes clear that al-Biruni indeed had the goal to enable Rayhana to become fully qualified to read research texts. This text is one of the very rare products of a medieval scholar of the mathematical sciences which surveys knowledge in a disciplinary sense. Even more, it is the only text I have ever seen in Arabic that teaches a beginner an understanding of all the disciplines united in the concept of the mathematical sciences with the addition of astrology.

[Question 1:] What is geometry?

[Answer 1:] It is the science of the magnitudes and the quantities in relationship to each other, the teaching of the properties of their forms and figures, as they pertain to a body. It transforms the science of the numbers from the particular into the universal and transfers astronomy from guesswork and opinions into truth. (Abu Raykhan Beruni 1975, 21)[2]

[Question 7:] What is a point?

[Answer 7:] If a line has an end, this end is a point. A point has one dimension less than a line: a line has length; a point has neither length nor breadth or depth. A point is the end of the ends; that is why it does not have parts. It is illustrated from among the sensible things by the head of the acute needle. Each one of the line, the surface, and the point exist in the body, which carries them. Outside of the body, one can only imagine them in the mind. (*Abu Raykhan Beruni* 1975, 23)

[Question 16:] What is the upright sine?

[Answer 16:] That is half of the chord of the doubled arc or, if you like, the perpendicular, which is placed from one of the two ends of the arc to the diameter, which is drawn from the other end of the arc. When you see "sine" free (from any qualification), know that this is the upright sine. (*Abu Raykhan* 1975, 24)

[Question 32:] How does one multiply a line with a line?

[Answer 32:] This is the procedure if one line is marked off on the other until a rectangular surface results, which those lines enclose. If the two are equal, then the mentioned surface is a square. If the two are different, it is an oblong (figure). (*Abu Raykhan Beruni* 1975, 27)

[Question 55:] What is an inverse proportion?

[Answer 55:] That is when the second and the third [magnitude] are on one side. That is obvious for loads of the steelyard, which is the *qabban*: the ratio of the distance from the cancer on it until the suspension is to the distance of the moveable counterweight until it like the weight of the counterweight to the load, which equilibrates it in the scale. (*Abu Raykhan Beruni* 1975, 33)

1 The Engish translation used here as a basis often modernizes the terminology in a manner that does not respect the philosophical theories of the period. I replaced such modernizations by more adequate terms, avoiding thus a language which is anchored in Newtonian physics.

2 I did not reproduce Ramsay Wright's English translation, which is not always correct: *The Book of Instruction in the Elements of the Art of Astrology*, by Abu'l-Rayḥān Muḥammad Ibn Aḥmad Al-Bīrūnī, translated by R. Ramsay Wright, London: Luzac & Co., 1934. But I compared the Arabic text of the facsimile reproduced by Wright, which I felt was often necessary. Hence, the translation of the extracts is my modification of the Russion translation on the basis of the Arabic text of MS London, British Library, Or 8349.

[Question 64:] How many figures can enclose a sphere?

[Answer 64:] If they have equal sides and angles, which are from one genus, then there are five only, which relate to the four elements and the celestial sphere from the side of similarity. But if they are composed from different kinds, then they are neither limited nor numbered. As for the first five figures, one of them is the cube with six square faces. It is called the earthy (one). The second has twenty equal-sided faces. It is the watery (one). The third has eight triangles (of the same kind). It is the airy (one). The fourth, the spiny (one), has four triangular faces. It is the fiery (one). The fifth has twelve pentagons as faces. (*Abu Raykhan Beruni* 1975, 36)

[Question 123:] What is the heaven?

[Answer 123:] The word "heaven" means everything that is above you and towers above you so that by restriction this word means the clouds and the roofs of the houses. In a free (sense), it is the ceiling that is visible to the world, which is the heavenly sphere whose description was introduced before. The Persians call it in their language *asmãn*, i.e. (something) similar to a millstone (due to?) its circular movement. (*Abu Raykhan Beruni* 1975, 51)

[Question 264: How many (periods) has a solar eclipse?

[Anser 264:] There are three, because there is no perceptable stay; they is nothing else than the beginning of the occultation, its middle, and the completion of the disappearance. (*Abu Raykhan Beruni* 1975, 125)

[Question 385:] Which are the male and which are the female planets?

[Answer 385:] The three upper panets and the Sun are male, whereas Saturn is a Eunuch [having no influence on the birth]. Venus and Moon are female. Mercury is a hermaphrodite, because it is male together with the male planets and female together with the female (ones). When it is, however, alone, it is male. Some consider Mars as female, but this opinion is not accepted. (*Abu Raykhan Beruni* 1975, 180-181)

[Question 491:] What are the "dead" degrees?

[Answer 491:] These are the five degrees before the degree of the ascendant in (the direction) opposite to the sequence (of the zodiacal signs). Ptolemy does not count them in the twelfth (house) and does not consider them as belonging to the horoscope. But if the planet is in them, then he considers them in the horoscope. (*Abu Raykhan Beruni* 1975, 239)

These few examples indicate that al-Biruni created a superb teaching document not merely by his comprehensiveness. He continuuously talks directly to Rayhana, explaining one matter, comparing the other, or offering a name in another language. The many terms, concepts, and possible difficulties are represented by diagrams and tables visualizing and ordering the taught knowledge. As a result, Rayhana, if she ever read the book from cover to cover, would have been capable of understanding fairly high-level scientific texts or participating in scholarly conversations, if allowed into the male circle. She could also have acted as a teacher to other women. But nothing of that aroused a historian's curiosity and hence remains hidden in the dust of history. One thing, however, al-Biruni did not teach her in his book: technical skills, ie. how to observe and measure the altitude, declinations, azimuths, and other coordinates of heavenly bodies, and to calculate values derived from them.

Although *The Book of the Stars* shows al-Biruni as a gifted teacher, no other teaching activities of his are known. But due to his transfer to Ghazna and the military campaigns into northern India, which he had to accompany, he engaged in his adult life in a series of learning activities.

5. al-Biruni's Acquisition and Distribution of Knowledge in India

Al-Biruni's learning of knowledge from non-Islamicate sources in India was part of his upbringing in the mathematical sciences as a result of the translation of Sanskrit texts on astronomical, astrological, and chronological subjets during the eighth and the ninth centuries and the integration of this Indian knowledge into arithmetic, astronomy, astrology, chronology, and to a limited degree geometry. Despite the overwhelming preference for translations of ancient Greek texts in most of these disciplines by professional experts, a good number of particulars from Sanskrit traditions remained within Arabic and Persian scientific knowledge practices.

During the first two decades in South Asia, al-Biruni learned Sanskrit and probably at least one of its spoken forms. He outlines fairly well the difficulties one can encounter in such an endeavor, although he does not name all of them. Then he went to look for teachers of philosophy, astronomy/astrology, arithmetic, and literature. He was upset about the socio-cultural ideas of the Brahmins who considered him impure and refused to interact with him. In his view, the Buddhists were not much more welcoming. But he himself harbored his own ideas of superiority:

> At first I stood to their astronomers in the relation of a pupil to his master, being a stranger among them and not acquainted with their peculiar national and traditional methods of science. On having made some

progress, I begann to show them the elements on which this science rests, to point out to them some rules of logical deduction and the scientific methods of all mathematics, and then they flocked together around me from all parts, wondering, and most eager to learn from me, asking me at the same time from what Hindu master I had learnt those things, whilst in reality I showed them what they were worth, and thought myself a great deal superior to them, disdaining to be put on a level with them. (*Alberuni's India* 1992, 23)

As alrady Sachau pointed out in his edition and English translation, al-Biruni does not provide any information about the men who taught him the language and any of the other kinds of knowledge found in his book. (*Alberuni's India* 1992, xxxv) Equally, he did not talk much about his learning experience. Only once does he state that it was very difficult for him to enter into Indian scientific doctrines, despite his love for the subject. He admits to have spent much time and money for buying books, even from remote places, and paying teachers. (*Alberuni's India* 1992, 24) Al-Biruni rather emphasizes that the goal in writing his book was to teach its readers. (*Alberuni's India* 1992, 110, 122, 147)

While the words teacher and teaching appear in al-Biruni's *India* mostly within stories about Indian gods, kings, princes, grammarians, the word learning is mentioned at least three times with regard to himself. In one instance, he quotes from a single page of a book by Brahmagupta (died after 665) about arithmetic, which arithmetical procedures he described for metric. (*Alberuni's India* 1992, 147) He expresses his hope to find one day the complete book, because much could be learned about Indian arithmetic from it. (*Alberuni's India* 1992, 150-151) In the chapter on weights, compiled so that the reader can better understand the various terms used throughout the book, al-Biruni briefly remarks that he learned about weights and coins and their equivalents with those used in his home region or by Muslim traders from some unspecified Indians. (*Alberuni's India* 1992, 160) This seems to suggest that he did not limit his acquisition of knowledge to the formal ways of learning with a teacher or a book. In the chapter on witchcraft, al-Biruni begins with discussing alchemy saying that the could not learn much from the Indian adepts of this art, which he himself does not think highly of. Nonetheless he managed to acquire some basic information about methods (sublimation, calcination, analysis, waxing of talc). On this basis he concludes "that they incline towards the mineralogical method of alchemy". (*Alberuni's India* 1992, 188) He also knew that the Indians practiced a science related to alchemy: rasayana, but that it was mostly plant-based and dealt with drugs and compound medicines. Further information provided by al-Biruni shows that he did not merely inquire about the content of a scientific discipline, but was apparently always eager to learn about its most famous practioners and their books. (*Alberuni's India* 1992, 189) This corresponds well with his approach in the chapters on astronomy, astrology, and chronology.

He decidated an entire chapter to the authors and books on astronomy, where he presents information acquired from Brahmagupta's *Siddhanta*, i.e. his *Brāhmasphutasiddānta*. ((*Alberuni's India* 1992, 152-159) He had begun translating it as well as the *Paulisha Siddhanta* (circa fourth century?), which he also had been able to buy. But when writing the *India*, he had not finished yet his translations. (*Alberuni's India* 1992, 155-156) However, he reports that he had finished his Arabic translation of Varahamihira's (505-587) smaller *Jataka* on astrology. (*Alberuni's India* 1992, 5) He also quotes repeatedly from Varahamihira's *Samhita*, having been obviously well familiar with his works. (*Alberuni's India* 1992, 162, 166-167) In addition to these three most often mentioned authors and their works, al-Biruni also quotes from commentaries by Balabhadra, who may have lived in the first half of the ninth century. This implies that his learning of Indian astronomy, astrology, and chronology rested mainly on books, some of which he may have read with a teacher. This is hinted at by al-Biruni, when he adds to the title of an astronomical handbook "which shows, as I am told, how the corrected places of the stars are derived from one another". (*Alberuni's India* 1992, 157)

In addition to books, authors, methods, terms, or parameters, al-Biruni also was very eager to collect stories. Across his entire book, he tells a great number of stories about all sorts of themes. One of these stories explains why Brahmagupta named one of his astronomical books *Karanakhandakadyaka*, where *khanda* refers to an Indian sweetmeat:

Sugriva, the Buddhist, had composed an astronomical handbook, which he called *Dadhi-sagraha*, i.e. the sea of sour milk; and a pupil of his composed a book of the same kind which he called K*uri-babaya* (?), i.e. a mountain of rice. Afterwards he composed another book which he called *Lavana-mushti*, i.e. a handful of salt. Therefore Brahmagupta called his book the Sweetmeat-khadyaka- in order that all kinds of victuals (sour-milk, rice, salt,

&c.) should occur in the titles of the books on this science. (*Alberuni's India* 1992, 156)

These stories as well as the level of technicalities in al-Biruni's book on India confirm his statement in the preface to the book that he wished to compile a reader-oriented book, not an introduction into the intellectual world of the Indians and its technicalities. The *India* was meant to provide information and entertainment, certainly important elements of learning. But it was not meant to teach specific, applicable knowledge in the styles of the various religious and scholarly communities of the subcontinent.

6. Conclusions

Studying extant mathematical, chronological, and historical works of Abu Nasr and al-Biruni does not lead to a complete picture of how the two boys learned and the two men taught. But it highlights specific features of their educational processes. Books played an important role in their understanding of how to acquire, produce, and teach scholarly knowledge. Reading and correcting books, checking their technical procedures, inquiring about practical implementations of methods and theories, challenging data and explanations, and exploring unknown cultures of knowledge are important components of Abu Nasr's and al-Biruni's lifelong practices of learning and sharing knowledge. Constructing instruments, carrying out observations and measurements, and cooperating with other scholars shaped al-Biruni's years of higher education, in all likelihood supervised by Abu Nasr. The cooperative relationships between al-Biruni and his colleagues portray him certainly as a highly gifted man and clever organizer of scholarly work. But they also speak of the respect Abu Nasr and others paid to him and the seriousness with which they participated in his investigations.

Al-Biruni's books on the mathematical sciences for Rayhana and on India for a broad range of readers show him as a gifted teacher with a clear eye for the scope of the sciences and an entertaining narrator with insights into foreign cultures of knowledge, but also limitations created by his own cultural identity.

Although some of these aspects look very familiar to us and thus speak for a shared intellectual bond over time, we should nonetheless abstain from demanding too much from them in terms of their similarity to our own practices and values. But with proper caution and respect for their own circumenstances and possibilities, their learning and teaching experiences as left to us in their writings can - I believe - inspire today's high school and university students.

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The Value of History in a Mathematics Classroom

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Abstract — The history of mathematics, used well, can be a powerful tool in the mathematics classroom. The historical process of mathematical discovery and verification shares many similarities with the process of student learning. We discuss the critical aspects of the learning experience that may be aided through the use of carefully chosen historical parallels, provide historical examples for these aspects, and consider some of the potential pitfalls.

Keywords—history; pedagogy; mathematics

I. INTRODUCTION

We've all experienced the problem. Given the mass of material we are required to cover in our math classes, it seems all but impossible to find avenues for creativity in our lectures. When small time windows open up, we tend to show extra problems, or new applications, or some favourite theoretical wrinkle that we had been saving for such an occasion. Why bring in history? It takes time and effort, and displaces other subjects. What's the advantage?

Simply put, history provides a path for the *entire* mathematical experience. Typically, our students are asked to solve problems and prove theorems, a limited part of what mathematicians do. The full story involves ***motivation:*** what is the context within which the subject arose, and why is it so appealing that it deserves our attention? Next is ***research:*** once the problem is identified, how do we articulate lines of attack that have already been made that might be adapted to the new situation? Third is ***critical thinking:*** how do we transition from received knowledge to new situations? Finally, we have ***implications:*** how does the solution affect us, the academic community, or society? Good history of mathematics synthesizes all these aspects. Bringing it into the classroom can provide for our students a much broader and deeper mathematical experience. Most crucially, history is a *natural* means to attain these goals: we follow real people, who struggled as our students do, and eventually (usually) triumphed. We learn best through stories, and true stories are often the best ones.

In the following, we provide examples of historical episodes to support each of these four aspects of mathematical development.

II. MOTIVATION

All mathematical subjects arose due to some need, either from within mathematics or from outside of it. These needs provide a reason to approach the subject, and to approach it in a certain way.

Example: Trigonometry was invented in ancient Greece to convert geometric models of the motions of the planets into quantitative predictions. The first trigonometric problem, likely solved by Hipparchus of Rhodes, was to determine the

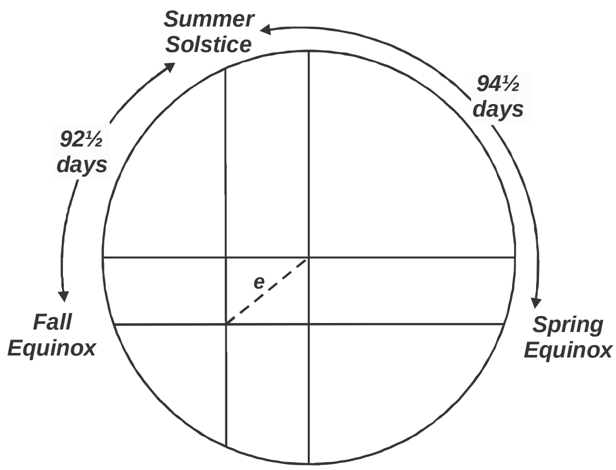


Fig. 1. Hipparchus’s model of the motion of the Sun.

eccentricity of the Sun's orbit around the Earth. In Figure 1, the Sun revolves around the orbit circle at a uniform speed; yet Hipparchus observed that the spring and summer are longer than the fall and winter. The goal is to place the Earth a certain distance and direction from the center of the orbit so that the lengths of spring and summer predicted by the model match their observed lengths. This distance e is the eccentricity of the Sun’s orbit. Hipparchus and his successors were aided in this calculation by a table, not of sines, but of chord lengths in a circle (Figure 2). The sine would be invented later, in India. Today, trigonometry is still a significant tool for moving back and forth between geometry and numerical measurement.[1]

[1] The earliest record, both of Hipparchus’s solar model (pp. 153-155) and of the calculation of a table of chords (pp. 48-60), may be found in Ptolemy’s *Almagest* [17]. A popular

Example: The need for a mathematical subject may not have been the same in the past as it is today. Logarithms, for example, were invented in the early 17th century by John Napier as a calculation device for astronomers to reduce the work

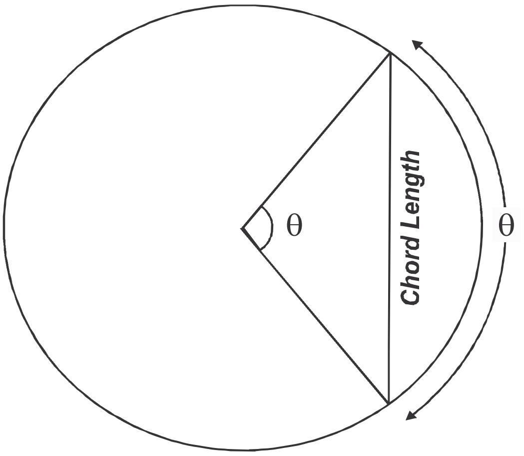


Fig. 2. The definition of a chord, the trigonometric function in ancient Greek astronomy.

Fig. 3. The title page of John Napier's 1614 announcement of his invention of logarithms, Mirifici logarithmorum canonis description.

involved in finding products and roots of numbers. (The cover of the book announcing Napier's achievement, *Mirifici logarithmorum canonis descriptio*, is shown in Figure 3.) Formulas in spherical astronomy, such as

$$\sin\delta = \sin\lambda \sin\epsilon$$

(which finds the Sun's declination δ from its longitude λ and the obliquity of the ecliptic ε), required astronomers to multiply or divide values of trigonometric functions, usually given to many decimal places, to solve for an unknown quantity. The use of the formula

$$\log(a \cdot b) = \log a + \log b,$$

accompanied by a table of values of logarithms, allows the astronomer to convert the problem of multiplication into the much simpler task of addition. This was such a boon that, almost two centuries later, Pierre Simeon de Laplace said, "by shortening the labours [logarithms] doubled the life of the astronomer." Today our computing power renders this use obsolete. Nevertheless, this historical route can provide a meaningful context for students' first exposure to the subject; the benefits of the theory are obvious, even if its original motivation is no longer active.[2]

III. Research

Coming to terms with methods that have been devised to

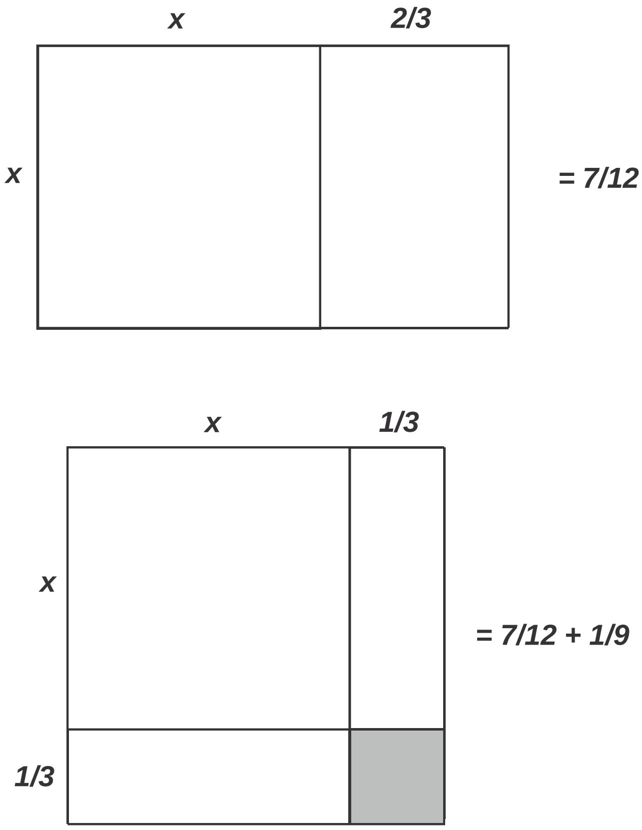


Fig. 4. A cut-and-paste method for solving a quadratic equation.

account of the chord table is in [1], pp. 121-126.

[2] Napier's *Descriptio* [12] is available on Google Books. See [9] for a treatment of Napier's logarithms adapted to the classroom (although not including Napier's trigonometry).

attack difficult problems is, by definition, a study in history. Techniques helpful in a historical situation may be applied elsewhere, and observation of successful processes of discovery can change a student's perspective on their own searches for solutions.

Example: Ancient Babylonian school children were already solving problems akin to quadratic equations. Although we do not have records of the original discovery of their methods, it seems clear that they must have been based on a "cut-and-paste" geometry. For instance, one such tablet solves the problem

$$x^2 + \frac{2}{3}x = \frac{7}{12}$$

using a computational prescription that mimics precisely the following argument (Figure 4): the left side of the equation can be represented as a square of unknown size appended to a rectangle measuring 2/3 by *x*; the combined figure is asserted to have an area of 7/12. Half the rectangle is cut off and moved to the bottom of the figure. The entire figure is filled in --- we "complete the square" --- by adding the small square in the bottom right. Then the area of the entire square is $\frac{7}{12} + \left(\frac{1}{3}\right)^2 = \frac{25}{36}$, so the side length is 5/6. Subtracting 1/3 gives $x = \frac{1}{2}$. Here we see that converting a numerical problem may gain new significance and an avenue to a solution when it is converted to a geometric arena.[3]

Example: Early in the history of calculus, it was realized that certain differential equations could be easily solved via the discovery that there exists a function that is its own derivative ($f(x) = e^x$). For instance, the standard differential equation for population growth,

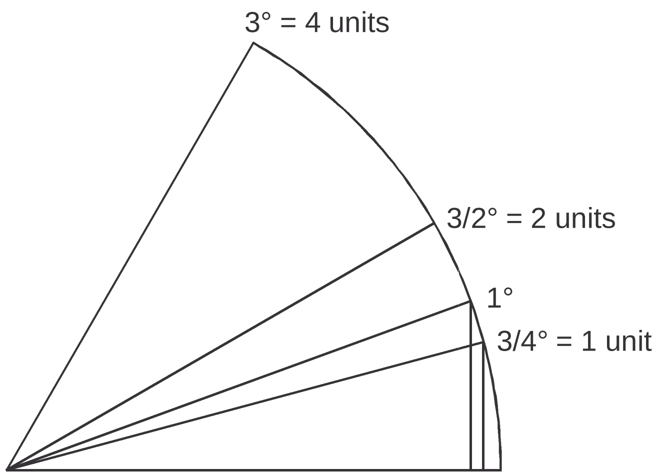


Fig. 5. Sines of small arcs. Most astronomers used degrees, which led to the unfortunate result that the sine of 1° (the larger of the two vertical lines) could not be computed precisely. Al-Samaw'al divided the circle into 480 units, so that the sine of 1 unit (equal to 3/4°) was attainable using geometric methods.

[3] A translation of this tablet and description of the student's method appear in [1], pp. 23-24. See [16] for an updated treatment of Mesopotamian mathematics in general.

$$\frac{dP}{dt} = kP,$$

may be seen to have solution $P = ce^{kt}$ by substituting into the differential equation. This method (taking a variant of the standard exponential function, inserting it into the equation, and altering the parameters to satisfy the equation) was exploited repeatedly to solve many problems from the 18th century onward. As mathematicians such as DeMoivre, Cotes, and Euler started to allow complex quantities to enter the equations through relations such as

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

differential equations representing periodic phenomena became accessible to the same method.[4]

IV. Critical Thinking

Every mathematical community makes shared decisions about the validity and power of various competing approaches. For instance, medieval Indian mathematics valued solutions that we might describe as approximate or iterative, while ancient Greek and medieval Islamic mathematicians preferred direct arguments and calculations. One of the most difficult concepts for modern students to understand is that such commitments are also present today. In order to think creatively, one needs to make informed judgments about alternate avenues of attack; one must know what the community's rules are before one decides to bend or break them.

Example: The 12th-century Iranian scholar Ibn Yaḥyā al-Samaw'al al-Maghribī, late in his life, composed a book entitled *Exposure of the Errors of the Astronomers*, in which he pointed out dozens of places in the works of his colleagues and ancestors where he perceived violations of correct mathematical practice, or downright mistakes. One of these episodes, the calculation of a table of sines, requires finding a value for $\sin 1^\circ$ from a known value of $\sin 3^\circ$. To find a precise solution turns out to require the solution of a cubic equation, and this implies that the problem cannot be solved with geometric methods --- it is equivalent to trisecting an angle. But trigonometry is applied geometry. Since Claudius Ptolemy's time, astronomers had been forced into approximate methods, violating the geometric spirit of the subject. Al-Samaw'al's creative solution was to redefine the number of degrees in a circle from 360 to 480. Then $\sin 3^\circ$ becomes the sine of 4 of the new units (see Figure 5), and applying the sine half angle identity to it twice yields the sine of 1 unit --- thereby bypassing the necessity of approximation altogether.[5]

[4] For a survey of Euler's mathematics, including much related to differential equations, see [5]. [6], pp. 268-281, focuses on Euler's contributions to the calculus. See [3] for historical reflections on the role of applications in the development of differential equations in the 18th century.

[5] [18] is an account of al-Samaw'al's novel approach; it also includes a demonstration that Samaw'al actually did not compute his sine table as he claims.


Bartholomæi
Pitiſci Grunbergenſis
Sileſij.
TRIGONOMETRIÆ
Siue.
De dimensione Triangulor:
LIBRI QVINQVE. item
PROBLEMATVM VARIORV.
nempe
Geodæticorum,
Altimetricorum,
Geographicorum,
Gnomonicorum, et
Aſtronomicorum:
LIBRI DECEM.
TRIGONOMETRIÆ SVBIVNCTI,
AD VSVM EIVS DEMON-
STRANDVM.
Sumbtib. et impens.
Dominci. Cusfodis
chalcograghi.
M.DC..

Fig. 6. Title page of the first edition of Bartholomew Pitiscus's *Trigonometriae* (1600).

Example: The cubic equation was solved in full generality by Gerolamo Cardano in 1545. However, in the case of certain equations such as

$$x^3 = 15x + 4,$$

his method produced nonsensical solutions like

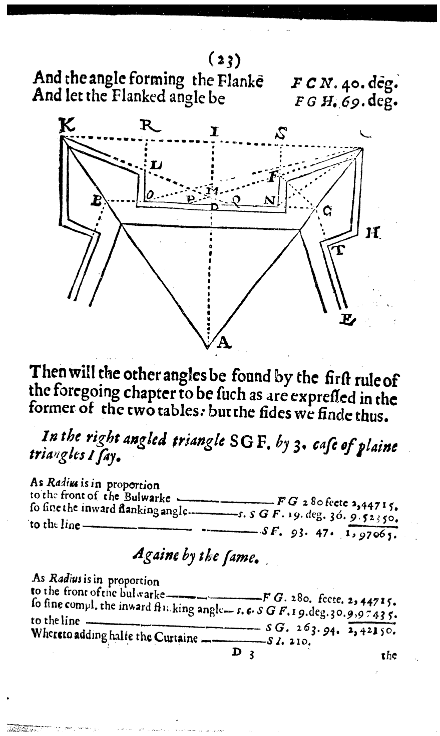
$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$

Rafael Bombelli, an accomplished algebraist, chose to break with Cardano and admit the possibility of the existence of square roots of negative numbers. He proceeded to develop the fundamental laws of complex numbers, and was able to demonstrate that the solution above reduces to $x = 4$.[6]

V. IMPLICATIONS

It is often said that the most powerful mathematical results are those that lead to new and interesting questions or that open mathematics to new applications. Witnessing the enlargement of the social role of certain types of mathematics can be a meaningful lesson in measuring its cultural significance.

Example: In early modern Europe the unification of trigonometry with logarithms in 1614 was motivated by the computational struggles of astronomers such as Tycho Brahe and Johannes Kepler. However, the new tool quickly found applications in a variety of disciplines. In the three decades before 1614, trigonometry had already been applied haltingly to altimetry (finding the heights of buildings and landmarks), surveying, and navigation. The title page of Bartholomew Pitiscus's 1600 *Trigonometriae* (see Figure 6), incidentally the title that coined the word, prominently refers to uses in geodesy,


(23)
And the angle forming the Flankē — *F C N.* 40. deg.
And let the Flanked angle be — *F G H.* 69. deg.

Then will the other angles be found by the firſt rule of the foregoing chapter to be ſuch as are expreſſed in the former of the two tables: but the ſides we finde thus.

In the right angled triangle SGF. *by* 3. *caſe of plaine triangles I ſay.*

As *Radius* is in proportion
to the front of the Bulwarke ——— *F G* 280 feete 2,44715.
ſo ſine the inward flanking angle ——— *s. S G F.* 19. deg. 36. 9.52350.
to the line ——— *S F.* 93. 47. 1,97065.

Againe by the ſame.

As *Radius* is in proportion
to the front of the bulwarke ——— *F G.* 280. feete. 2,44715.
ſo ſine compl. the inward flanking angle— *s. c. S G F.* 19. deg. 30. 9,97435.
to the line ——— *S G.* 263. 94. 2,42150.
Whereto adding halfe the Curtaine ——— *S I.* 210.
D 3 the

Fig. 7. A page from Richard Norwood's *Fortification, or Architecture Military* (1639).

altimetry, geography, the theory of sundials, astronomy, and (in a later edition) architecture. This was a revolutionary contact of mathematics with the physical world. The introduction of logarithms accelerated the acceptance of mathematics into practical domains. Figure 7, for instance, shows an image from Richard Norwood's 1639 *Fortification, or Architecture Military* demonstrating how to construct military bases using the new computational science. In a very short time, mathematics had moved from the Platonic realm and from the heavens down to the earth. Through this episode, mathematics was transformed into an engine that eventually helped to reshape modern culture through science and technology.[7]

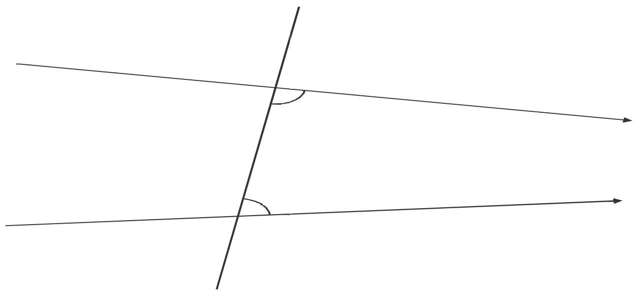


Fig. 8. Euclid's fifth postulate asserts:

[6] [11] is an accessible history of the birth and rise of complex numbers. For a classroom unit on Cardano and Bombelli, see [4].

[7] Pitiscus's *Trigonometriae* [15] may be found on Google Books; Norwood's *Fortification* [13] is in Early English Books Online.

Euclid's fifth postulate asserts: (fig 8) if the two indicated angles are less than two right angles, then the two lines with arrows, if extended, will meet on the side of the arrows.

Example: From the beginning of geometry until the past two centuries, it was assumed (both implicitly and explicitly) that space is Euclidean --- that is, that Euclid's parallel postulate holds. This asserts that, in Figure 8, if the two indicated angles add to less than two right angles, then the two lines meet in the direction indicated by the arrows. This turns out to be logically equivalent to the statement that the sum of angles in any triangle is 180º. After dozens of failed attempts to prove these claims, three separate mathematicians in the early 19th century (Carl Friedrich Gauss, Nicolai Lobachevsky, and János Bolyai) began to consider the possibility that the parallel postulate may be false --- or, at least, that a new and consistent geometry may be born of the assumption of the postulate's contradiction. Through this bold step, they created "worlds out of nothing", elliptical and hyperbolic geometry. These non-Euclidean geometries remained creations of the mind for decades. However, starting in the early 20th century they have become candidate models for the geometry of the universe in which we live. Also, significantly, this episode has revealed that seemingly obvious assumptions that survive for millennia (whether in mathematics or elsewhere) are not necessarily on solid ground.[8]

Students aware of cosmic shifts such as these participate in a much richer educational experience. They are better able to orient themselves and their chosen discipline in a significant place in the intellectual landscape, and are able to act in their profession with more reflectiveness and purpose.

There is one additional aspect of mathematical work that history can support: ***communication***. Since history encompasses entire narratives from initial conception to final product and societal impact, there is a unique opportunity here to improve students' ability to write and otherwise present ideas. Students can write essays; they can make presentations on the background and significance of sub-disciplines; they can write short-answer responses to questions about the significance of and interconnections between theories. It is usually difficult to find opportunities to improve mathematics students' rhetorical skills; history provides a powerful solution.

VI. CHALLENGES

Although the potential benefits of history are diverse, several dangers must be avoided.

a. Misunderstanding history as mere biography

Textbooks often give snapshots of mathematicians' lives and works in the page margins, mistakenly believing they have done a service to history. They have not. Many of these biographies are unrelated in any direct way to the narrative in the text, and so they unintentionally reinforce the tacit misconception that the mathematics itself is ahistorical. Genuine history in the classroom should be *part* of the presentation of the mathematics; its benefits can only be realized with deeper integration.

b. Entering history without sufficient depth

The history of mathematics is a deeply challenging endeavour, requiring sophistication in two disciplines with very different aims and modes of thought. Unfortunately, not everything one finds in the library or online is reliable, either historically or mathematically. The mathematics teacher should consult reliable sources; looking up reviews in professional journals is an effective way to screen out low-quality content.

c. Assuming that history is a universal panacea

Although history is helpful in learning many mathematical concepts, assuming that it *always* leads to positive results is dangerous. Choose moments where the historical context genuinely interacts with the subject, and is appropriate to students' concerns and maturity levels.

VII. PLACES TO START

For topics in the undergraduate curriculum, there is no better place to begin than Victor Katz's history of mathematics textbooks [10]. For accuracy, mathematical rigour, and thorough coverage, they are unsurpassed; and they provide many connections to the rest of the literature. Also consider a new book to be released in 2016 at MAA Press, based on the history of mathematics course at the Open University in the UK. At an elementary level, consider William Berlinghoff and Fernando Gouvea's *Math Through the Ages* (2nd edition) [2]. Finally, the *MAA Notes* series has published a number of volumes of historical episodes ready for classroom use, edited by Victor Katz, Amy Shell-Gellasch, Dick Jardine, and others (see for example [4], [8], and [9]).

Many modern theories of education attempt to address the plague of passivity in our students by promoting active educational experiences, such as the Moore method and inquiry-based learning. History provides the kind of engagement these innovations attempt to foster. However, history can also enhance the traditional mathematics classroom. By considering the entire cycle of mathematical development, and by asking students not merely to perform calculations but also to reflect upon them, history makes students more powerful, more thoughtful, and more significant. In short, it makes them better mathematicians.

ACKNOWLEDGMENTS

An early outline of this paper appeared in the *Canadian Mathematical Society Notes*; my thanks to the CMS for permission to reprint it here. Ariel Van Brummelen provided valuable help with respect to formatting and diagrams.

[8] [7] is a textbook on the rise of non-Euclidean geometries aimed at the upper undergraduate level. [8] provides a summary of non-Euclidean geometry, its impacts on physics, philosophy, and art, and its role in the classroom.

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BASED INCREMENTALITY MEASURMENT PARAMETERS OF CROSS-DOMAIN METHODOLOGIES

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ABSTRACT
This paper presents methodology for the development of complex real-time systems. The methodology uses incremental mechanisms in order to establish measurement parameters of cross-domain development process. It is an important objective to develop a cross-domain framework that addresses the requirements and constraints of several kind of industrial models. By promoting a strict increment-based design style and identifying increments that can be deployed in different application domains, the design and production costs of new applications can be significantly reduced by reusing components.
We will provide in this paper an overview of parameters measurement of cross-domain development process based on MARTE (Modeling and Analysis of Real-Time and Embedded Systems) and SART (structured analysis real time) using two industrial case studies for development of hydrostatic bearing lubrication system and triaxial apparatus system .

Key-Words: *Incremental approach, cross-domain development process, SART, MARTE, hydrostatic bearing, triaxial apparatus test, parameters measurement.*

1. INTRODUCTION

Along with technological advances, new needs arise in an industrial application and real-time **[7]** induced to add new features keeping up technology deployment already achieved.

The efficiency of any system depends on external and / or internal constraints which it is subjected. Two solutions can be envisaged in this respect to increase this efficiency: either we shall replace the system with a more efficient or improving it. The incremental method embodies the latter. It gives a framework for the development of systems as it integrates tools providing an environment of organic development. Each development in addition to improving the existing system enriches the while avoiding the costs of a new business solution. In addition, staff training time is reduced. As example we find based component technology **[13]** which uses software pieces called component. These components can be used independently or linked to construct new components or new packages. Industrial and academic community interest at first to components models of packages like EJB, CCM et .NET **[15].** Component field applying was extended to low level layers such systems and inter-systems **[13].** The general model of development process was suggested in Across project **[16].** In this kind of project, BIP (behaviour interface process) formalism **[12]** was used to capture behaviour of the studied system and tools are used to translate him to executable platform.

All these platforms need measurement parameters in order to evaluate meeting performance meeting like reliability, maintainability, survivability, etc. **[14].** The objective of this research is to develop a general methodology for designing a real-time application. The methodology is based paradigms such as MARTE, SART **[1,7,17]**. The concept applied uses the notion of incrementality **[2,3,11]**.

In this paper we present the idea based on instrumentality notion in order to measure performance of cross-domain methodologies: two case studies will be studied in the industrial fields: first, rotating machinery is commonly used in many mechanical systems, including electrical motors, machine tools, compressors, turbo machinery and aircraft gas turbine engines. Typically these systems are affected by exogenous or endogenous vibrations produced by unbalance, misalignment, resonances, material imperfections and cracks **[4]**. To damp the vibrations has been proposed several methods: passive **[10]**, active **[8]** and semi-active method **[9]** . Where passive methods are not enough to dampen the vibrations generated, a new smart hydrostatic journal bearing with four hydrostatic bearing flat pads fed by electrorheological fluid, has been designed to control rotor vibrations caused by imbalance and to reduce transmitted forces to the bearing. **[6]**. So bearings are machine elements used to guide the rotating shafts **[5]**. The hydrostatic bearings can be used irrespective of the load and speed. They are used successfully in a large number of machines operating at low speeds and carrying heavy loads. Lubrication of hydrostatic bearings is an important process, but also complex. Because it consists of electronic and mechanical components that operate in a physical world. This is what makes the development process is also difficult.

The second industrial domain consists of test triaxial apparatus. The triaxial test is one of the most versatile and widely performed geotechnical laboratory tests, allowing the shear strength and stiffness of soil and rock to be determined for use in geotechnical design. Advantages over simpler procedures, such as the direct shear test, include the ability to control specimen drainage and take measurements of porewater pressures.

The idea is to apply an incremental approach for incremental specification, and we try to adopt the method of SART, MARTE specifications an incremental way.

This paper is organized as follows: sections 2 present the incremental approach and our proposition, Section 3 shows our case studies with a specification of the SART and MARTE methods Finally, we conclude with discussion how measurements parameters could applied to the case studies.

2. METHODOLOGY

2.1 Incremental approach

The objective with this model is to quantify parts which can be developed from specification to executable code. The development of an increment may follow several approach for example a spiral approach. Incremental development means dividing the requirements into suitable parts during the specification allowing for independent development of the different increments **[3].** The design and coding of one increment are followed by testing of that increment, which makes it possible for the developers to start implementing the next increment while the testers validate, verify or certify the first developed increment. The incremental approach hence allows for a good deal of parallelism between development and testing. The benefit from this parallelism is not only the possibility to work in parallel, but also that the testers really start testing the software to be delivered at an early stage. We propose an incremental approach to the specification stage system based on the methods of specifications such as SART, MART. **[18]**... Figure 1 illustrates the process of our methodology.

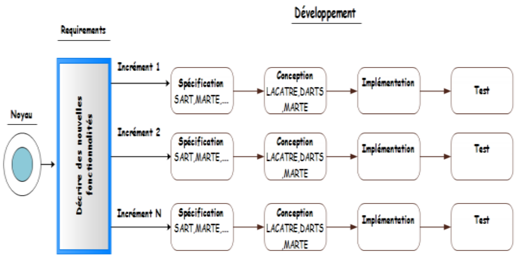

Figure 1: Incremental development approach

It will define the development of an application from first abstract sketches to detailed integrated models that can be verified, simulated and deployed. Key to development process is the use of library of predefined services, which is available to the application developer. These components serve as building blocks for new applications et can be interfaced from a domain specific language. Applications are developed in specific descriptions language (MARTE, SART) **[19]** and translated to the general model, which captures its behaviour in tool language as well as the interfaces and connections. The general model is the starting point for model to model transformation to generate the inputs for later analysis and deployment steps.

In this section, we provide a brief overview of the IK methodology, which aims to develop novel model of life cycle metric, based on incremental model depicted in Fig.1, to improve existing practices in development of complexes real-time and embedded systems. The methodology consists of three essential elements: assets activities (incremental model), New activities (K model), and tools (with the methods and techniques used in each activity). Initially, the left-level system design models are carried out using the Incremental model. When the developers execute the second increment the k model is starting, the third increment execute with IM model, then the K model therefore is carried out, until N increment the IM and K model are affected with consecutive manner.

Algorithm for IK methodology:

I=1; IM
For i=2 to N do
Begin
IM;
KM;
I=I+1;
End

The main activities of this IK methodology are: IM model:

The classical stages for incremental model: Specification, Design, Implementation, and Test.

K model:

1. Collection increments Specification
2. Design
3. Implementation
4. Test

2.2 Measurement parameters

Among metrics applied to software systems contain the following elements: (a) reliability (b) survivability and (c) maintainability. Reliability is a function of how that customer will use the software. Reliability is determined by the interaction between the structure of the code and the user's operation of the system. The **survivability** can identify potential problems as they occur and seek remediation for these problems before the system fail. A system based on principles of survivability will be able to identify new usage patterns by the customer and communicate these new uses to the software developer.

A **maintainable system** is one that is built around the principle of requirements traceability. if it becomes necessary to change the system requirements, this becomes an impossible task when we do not know which code modules implement which requirement. Basically, a maintainable system is one that can be fixed or modified very quickly. These three metrics can be performed using the notion of increments.

3. CASE STUDIES

3.1 HYDROSTATIC BEARING

3.1.1 Presentation

A pump supplies a bearing about 30 % higher than that required flow rate (See figure 2). The excess fluid returns to the reservoir via a pressure regulator. A pressure sensor is used to stop the rotor drive if the pressure reaches a value too low. Non-return valve and the hydraulic accumulator provide food bearing to a stop of the shaft. We can also provide a backup pump. The flow is then derived to each cell on each portion of the circuit. Provision may be a check valve in case of overpressure in a cell. Resistance hydraulic HR should be placed as near the cell to avoid instabilities due to the pneumatic type lubricant compressibility. A pump may be necessary to ensure the return of the lubricant to the reservoir. A thermocouplee to control the temperature of the liquid at the outlet of the bearing and 1 trigger stop if it becomes too large. Finally, a cooling system ensures a constant temperature on the power supply **[6]**.

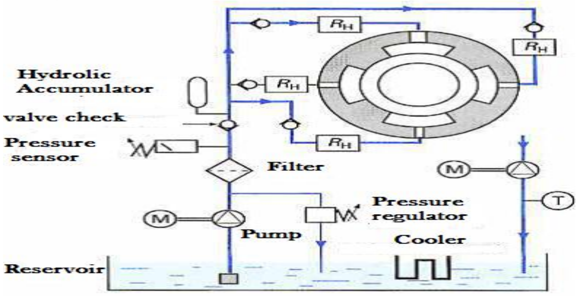

Figure 2: operating principle of hydrostatic journal bearing with four hydrostatic bearing Supply constant pressure scheme: real case

3.1.2 System specification

- Context diagram (figure 3)

First data flow diagram permit to describe the application environment.

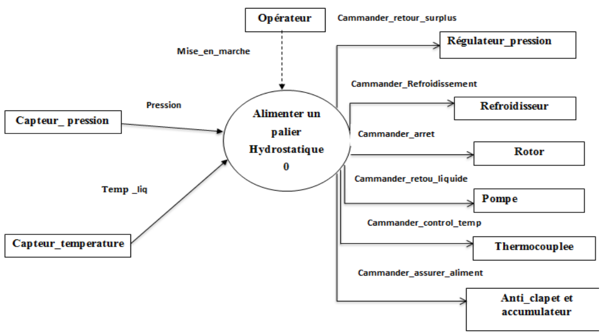

Figure 3: Context diagram

-Preliminary diagram (figure 4)

Data flow diagram showing the first level of functional analysis of the application.

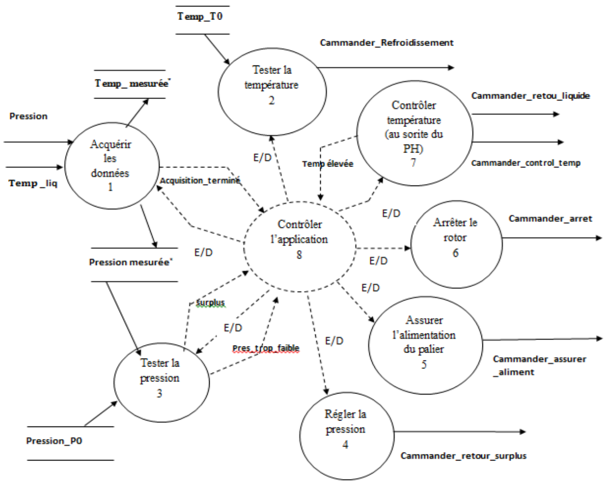

Figure 4: diagram preliminary

For this system we must add two features: (figure 5)

1. One feature to verify the level of fluid reservoir.
2. A feature to manage all previous functionality it functions as an operating system.

Check the liquid level in the reservoir according to our approach is an increment becomes a functional process To achieve this increment, we need to a sensor sends a signal when the low level (eg level b), the sensor is considered a terminal send a data level low = true or level low =false, finally needs an actuator here is that the pump will start.

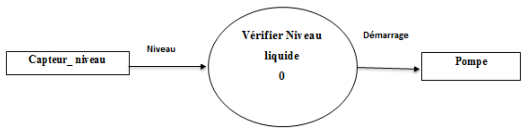

Figure 5 Diagram context (increment)

We now describe the preliminary diagram (Figure 6) :

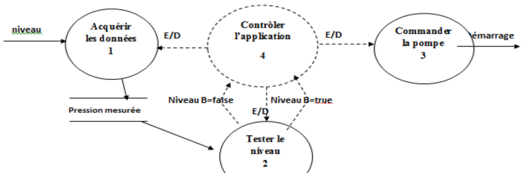

Figure 6: Preliminary diagram (increment)

- **Simulation and test**

Functionality to manage the various previous functions this feature can be illustrated by either a soft or hard process.

Our choice is to represent this functionality with RTOS which illustrate in the following figure 7 corresponding to hardware elements(figure 8).

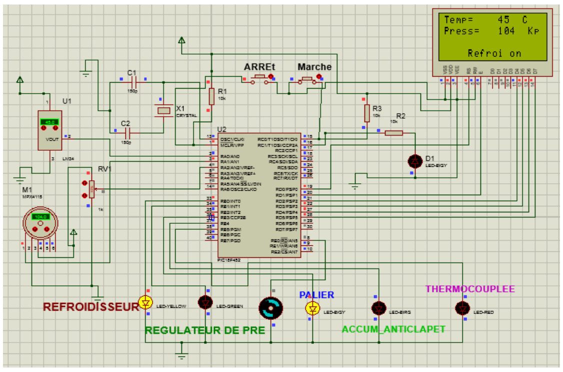

Figure 7: C-Micro-controller simulation

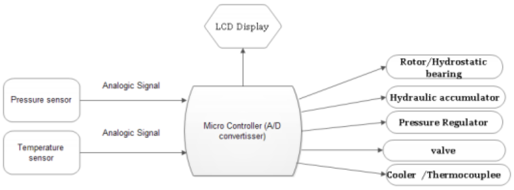

Figure 8: Diagram of hardware requirements

3.2 TRIAXIAL SYSTEM

3.2.1 Presentation of the system

The first stage of any strategy of construction is to determine if the ground of site concerned can accommodate the project. Thus, one is obliged to test and identify the properties of the ground of this site and to determine his capacity to support the structure. To arrive at an optimal solution, one analyzes in laboratory a sample of ground following of the methods by using a given apparatus. In this study, we chose the triaxial apparatus **[20].** The triaxial compression test makes it possible to better reach the mechanical properties of materials **(Figure 9)**, because it affects the state of in situ stresses. This type of test allows to control and measure the pore water pressure and to apply a range of confining pressure (isotropic or anisotropic) to initially consolidate the sample in a preset state. The various realizable types of test are:

- Test UU (Unconsolidated-undrained): test unconsolidated not drained carried out on saturated material or not
- Test CU (Consolidated-undrained): consolidated test not drained on saturated material or not
- Test CU+u (Consolidated-undrained): consolidated test not drained on material saturated with measurement of the pore water pressure.
- Test CD (Consolidated-drained): consolidated test drained on saturated material.

It is essential to specify which cohesion C and which angle of friction Φ are determined by the triaxial compression test, in particular for the fine grained soils for which C and Φ dependent degree of saturation, rate loading, field of consolidation in which the triaxial compression test is carried out. **[9]**

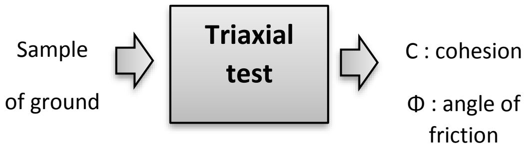

Figure 9: Objective of the triaxial test

3.2.1 System specification

We used the diagram of sequence to represent graphically the interactions between the actors and the system according to a chronological order. We used also the diagram of components to describe the organization of the system from the point of view of the material elements like the sensors (pressure, force, displacement and change of volume), and the other various components of the triaxial apparatus in order to highlight the dependences between the components.

- Sequences diagram (Figure 10)

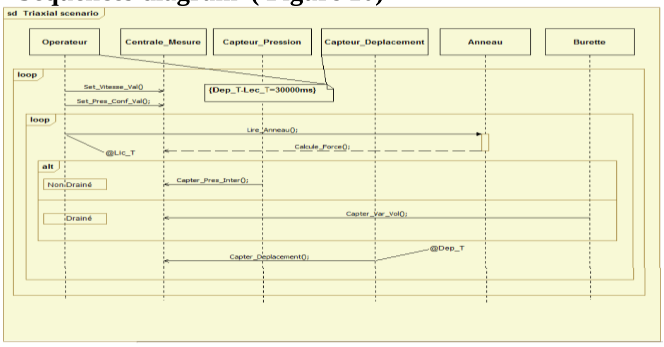

Figure 10: Sequences diagram

The operator must put two parameters (rate of displacement and confining pressure) and he must each 30 second read the various sensors. The force is calculated via value of the deformation of the ring, which we raised starting from the gauge According to the type of test, the operator take the value of pore water pressure if the test is not drained either the value of variation of volume if the test is drained. Displacement is also measured.

- Components diagram

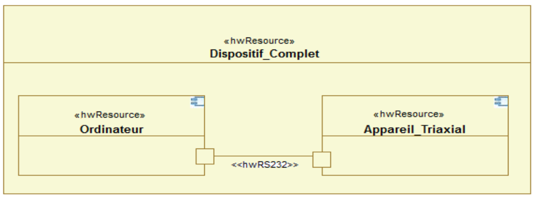

Figure 11: Model interns of the complete device

We notice at the beginning that the complete device is composed of two principal parts, which are the "Computer" and the "Triaxial Apparatus" which are connected through a "CableRS232" **(Figure 11)**.

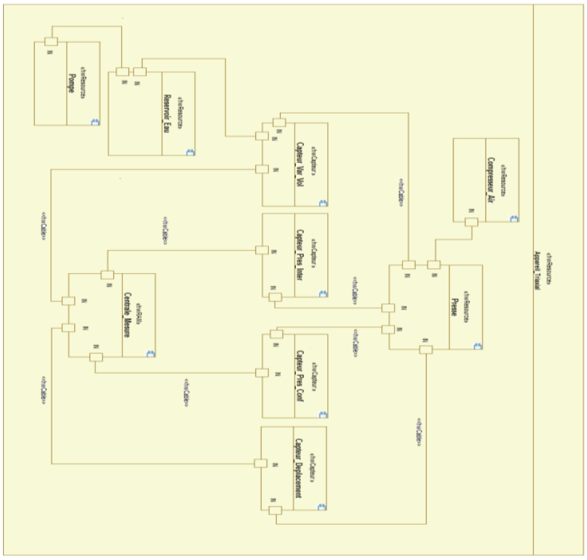

Figure 12: Model of component of the triaxial apparatus

Triaxial apparatus is also composed of "Power station of measurement" and the various "sensors" which are also connected to the power station of measurement. One also finds the "Compressor of the air", "Water tank" and the "Pump" **(Figure 5)**.

- **Simulation and Tests**

The triaxial device is equipped with a datalogger (Figures 13 and 14) connected to the computer by RS232, the datalogger has four inputs for connecting sensors of the device.

Figure 13: Front capture of datalogger

Figure 14: Behind face Capture of datalogger

Applying the notion of incrementality, we proposed to change the strategy of the manual measurement of force by another automatic technique replacing the strain gauge with a digital displacement sensor. This technique helps us to calculate the force from the deformation of the ring and the amount of movement made by the new sensor. At this point, we can say that the system becomes more automatic. For this we changed the operation of the datalogger using our map (Figure 15) keeping the sensor inputs and power and replacing the strain gauge with the new digital displacement sensor.

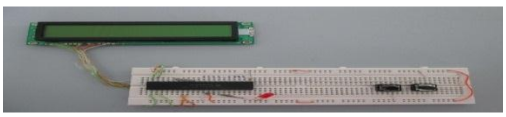

Figure 15: New realized circuit

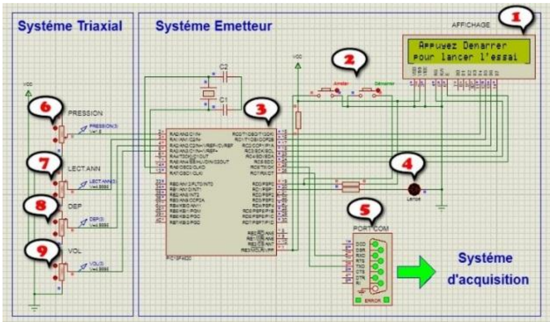

Figure 16: C-Micro-controller simulation

1: LCD display , 2 : Pushbutton Control , 3: PIC18F4520 Microcontroller , 4: Lamp , 5: Port COM , 6: Pressure sensor , 7: Force Sensor , 8: displacement sensor, 9: variation of volume sensor

When launching the simulation system (Figure 16), a message appears on the LCD "Triaxial Test CONRO LAB" for two seconds and then the LCD displays the message "Press start to start the test". After pressing the "Start" button, the LED lights and the LCD displays "Data acquisition". The system starts to read the sensor values as voltages and send them to the COM port to acquire in acquisition software. At the end of the acquisition, the LCD displays a message "Test Complete". If we press the "Stop" button, the system stops and starts.

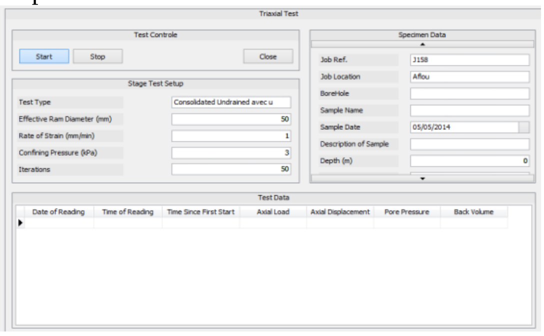

Figure 17 : Starting test window

Figures 17 and 18 present consolidated essay not drained, three essays associated to pressure curves.

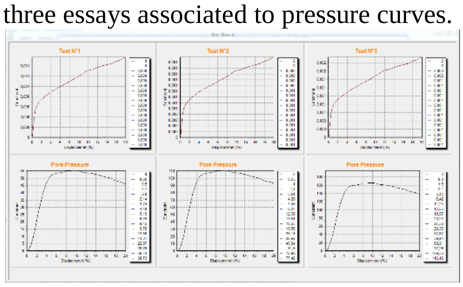

Figure 18 Consolidated and not drained essays

4. DISCUSSION & CONCLUSION

In this paper, we proposed an approach incremental in order to establish measurement parameters performance of applied methodologies for application real time systems.

	Tools	activities	Metrics
Case study one			
Specification	SART : Functionnalities	Data Flow Diagrams (DFD)	Reliability
Design	mapping functionnalities to DFD	Increments Computing	Reliability
Implementation	Mapping DFD to software Modules	Increments Computing	Survivability/maitenablity
Test	Mapping software Modules to Hardware Modules	Increments Computing	Survivability/maitenablity
Case study two			
Specification	MARTE : Functionnalities	Diagram of Components (DC)	Reliability
Design	mapping functionnalities to DC	Increments Computing	Reliability
Implementation	Mapping DC to software Modules	Increments Computing	Survivability/maitenablity
Test	Mapping software Modules to Hardware Modules	Increments Computing	Survivability/maitenablity

Table 1 : Correlation between life cycle software and software engeeniring metrics.

Table 1. Illustrates correlation between software engineering metrics and life cycle of specific domain applications (geotechnical and fluid mechanics) using based increment notion. This approach of measurement can be extended to other software engineering metrics and be applied to other methodologies like Across ARTEMIS projects etc.

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Improvement of The Lifetime of Wireless Sensor Network State of the Art

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Abstract
Thewirelesssensor networksare of considerable interestand a new stagein the evolution ofinformation technologyand communication. This new technologyis a growing interestgiven the diversityof these applications: health, environment,industry and evenin sports. Unlike traditionalnetworks, whichare concernedto ensure goodquality of service,wireless sensor networksmust, in addition, take into account theconservationof energy.They mustincorporate mechanismsthat allow usersto extend thelife of theentire network, since eachnodeis powered by alimited energy sourceand usuallyirreplaceable.In a sensornode, energy is consumedby providingthe following functions:capture,calculation(treatment)and communication.The latterrepresents a largeportionof the total energyconsumed.Therefore,the research communityis beingdeveloped and refinedseveral techniques forenergyconservation.Inthispaper, we presenta state of theartfor improvementthelifetime of wireless sensor networks.

Keywords:*Wireless sensor networks ; Energy efficiency; Power management ; sensor network lifetime ; energy Harvesting*

I. Introduction

Sensor networksrefer towireless networksspontaneous(structure emerging from interactionbetween nodesaccording to the principlesof self-organization and self stabilization)dedicated to the observationof complex dynamicphenomena: Thesensor nodesaresmall autonomouselectronicdevices, battery powered andequipped withwirelesstransmission capacity(commonlyradiowaves), which will measurephysical quantitiesandcooperate topass the information, step by step, to acollection point.Sensor networkshave obviousadvantages:they can be deployedveryquickly, coververy largegeographical areasandoperate withouthuman interventionwith highfault tolerance.

Because of theirflexibility, low costand easeof deployment,wirelesssensor networkspromise to revolutionizeour livesthrough severalapplication areassuch assensing anddisaster monitoring, control of theenvironmentand mappingbiodiversity,intelligent building, precision agriculture, monitoring and preventive maintenance ofmachinery, medicine and health,logistics andintelligent transport.

Forlong-term monitoring, that is to sayfor networkswhoseexpected lifetimeisof the order ofseveral monthsor years, theconservation of energyis a fundamentalproblem given thatthe replacementbatterysensoris usuallyimpossible.The stressenergymotivatesmany of theresearch problemson networksofwireless sensors. La consommation d'énergie des capteurs joue donc un rôle important dans la durée de vie du réseau qui est devenue un critère de performance prédominant.

Several researchrevolvearound a commongoal: the identificationand characterizationof the mostenergyconsuming activitiesand optimizingthe energy consumption ofnodes-sensors. It is commonlyaccepted thatthe radio transmitter isone of the mostenergyintensive components[1] [2]. Thusmostofthe energy dissipated byasensornodefor the transmissionandthedata reception.

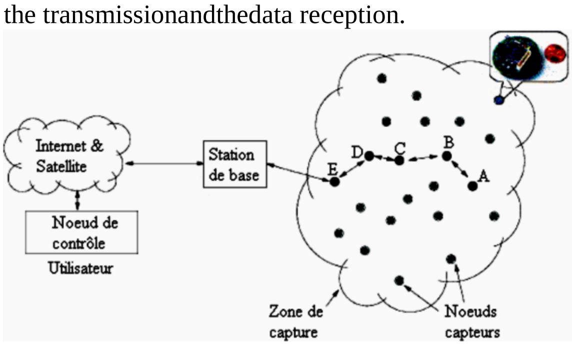


FIG.1.Sensor network architecture.

In this article our objective is to present a state of the art concerning improving the lifetime of wireless sensor network. The rest of this article is organized as follows: Section 2 briefly describes the various sources of energy in the external environment recovery. Section 3 presents the existing energy conservation techniques used in wireless sensor networks.

II. HARVESTING ENERGY FROM THE EXTERNAL ENVIRONMENT

Theenergy sourceiseasier to usethe battery.In the caseof the sensornode in awireless sensornetworkthat solutionis unacceptable becauseitbecomesinactivewhen the batteryis exhausted. In other wordsto increasethe autonomy of thesensornode, it isnecessary to increase thesize of the battery, which is inconsistentwiththe authority toautonomousmicrosystemintegration.The solution thenisto use rechargeableenergy reservoirsornamed assecondarybatteries. Themost common sidetanks arerechargeablebatteries.Currentlyconsiderable effortsin this area aremade toco-integrate these energysources.Thesehighenergytanksneed aprimary energy sourceforrecharging.In the following wewill brieflypresent someprimary energy sources.

A. SOLAR ENERGY

Systems usingsolar andlight energyare most commonlyregardedas a source ofenergy recovery.Photovoltaic cellsoffer excellentpower densityin adirect exposure tosunlight.The light energyavailable insideishoweversignificantly reduced.[3].

B. ELECTROMAGNETIC WAVE

Another approachis to harvestthe powerof propagation ofRFsignals (radio frequency) emitted by strong electromagneticfieldssuch astelevision signals,thecell phone towersandradio waveswireless networks(GSM,Wi-Fi).TheRFenergy can beextracted from the airandconvertedinto a DC voltageusablebyaconverting circuitincorporated in apowerreception antenna. However, the levelofrecoverable powerislimitedbecause theRFenergyspreadrapidly decreasesas thedistancefrom the source increases.

C. THERMAL ENERGY

Thetemperature gradients existingin natureand the human bodyhasthe capacity togenerate electricity. The most basicthermoelectric generatorcomprises athermocoupleconsistingoftwodifferentblocksconnected bya metal stripmaterial. The temperature differencebetween the lower partandthe upper partof the legsresults inan electrical current(Seebeck effect). Generally, the jambes are made of highly doped semiconductor p and n type for better performance, combining good electrical conduction and heat resistance (to keep the temperature gradient). Practically, athermoelectric energyrecoveryisthermopilethatis constitutedofmanythermocouplesconnected electrically inseriesandthermallyin parallel.Mostresearch focuseson the optimization ofnanostructuredthermoelectric materialsand their geometryto produceenough power andvoltagefromtemperature differencesas reduced of5-10 ° C. One challenge isto maintainthe temperature gradientbetween the hotand coldsmall scaleregion, includingwhen usedon the human body [4].

D. L'ÉNERGIE MÉCANIQUE

One method ofharvestingthe most effectiveenergyis to convertmechanical energy frommovement orvibrationsinto electricalenergy using, for example, electromagnetictransducers,piezoelectricorelectrostatic . Conversion istwomechanisms: direct applicationof force andthe useof the inertiaacting onamassforces.

Electromagnetictransducersare difficult tominiaturize, tohaveamicroscopic scaleofvery highimpedanceloadandgenerate onlylow voltages.Electrostaticgeneratorsrequire for theirshareof a potentialapplicationon startup anddeliveralow power density, but they have the advantage of beingmore easily integratedwithCMOStechnology(Complementary MetalOxide Semiconductor).

Piezoelectric recuperators have in turn the advantage of greater power density generated in a small scale [3].

III. ENERGY CONSERVATION SCHEMES

Experimental measurementshave shown that, generally, it is the transmissiondatawhich is the mostenergyconsuming, andsignificantly, the calculations, theyconsumeverylittle [5]. Theenergyconsumptionof the detection moduledepends on the specificityof the sensor. In many cases, itis negligible comparedto the energy consumedbythe processing moduleand, above all, the communication module. In other cases, the energy usedfor the detectioncan be comparableorgreater than thatrequired for the transmissionofdata. In general,energysaving techniquesfocuson two parts :the network portion(ie,power managementisreflected inthe operations of eachnode, as well as in the design ofnetworkprotocols), and the detectionpart (ie, techniques are used to reducethenumber or frequencysamplingtheenergycost). The lifetimeof asensor networkmay be extendedby the combinedapplication of differenttechniques [6].

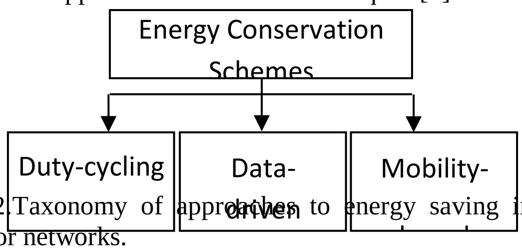
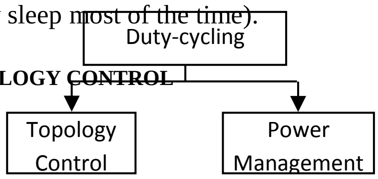


FIG.2 Taxonomy of approaches to energy saving in sensor networks.

III.1. DUTY-CYCLING APPROACHE

This techniqueis mainly used innetwork activity. The most effectivewayto conserve energyis toturn the radiotransmitterin(low-power) standby modewheneverthecommunicationis not necessary. Ideally, theradiomust be turned offas soon as itis no moredata to sendandor receive, and should be readyas soon as anew data packetto be sent orreceived. Thus, the nodes alternate betweenactiveandsleepperiodsbased onnetwork activity. This behavior is usuallyreferred to asDuty-cycling. Aduty-cycle isdefined as thefractionof time thatthe nodesare active.Asnodes-sensors performtasks in cooperation, they must coordinate theirdatesof sleep andwaking. ASleep /Wakescheduling algorithmthereforeaccompany anyplanDuty-cycling. This is usuallya distributedalgorithmbased onthe dates on whichnodesdecide toswitch betweenthe active stateandsleepstate. It allowsneighborsto besimultaneously activenodes, which enables the exchangeof packets,evenif the nodeshave a lowduty-cycle (ie, they sleep most of the time).



III.1.1. TOPOLOGY CONTROL

The topology control is to eliminate unnecessary network nodes (in the sleep redundant nodes) and links unnecessary (by adjusting the power of the radio transmitter, and therefore the scope of the communication) to reduce the energy consumption in the network. It istherefore toareduction of the initialnetwork topologywhile preservingthe coverage ofthe area of interestandnetwork connectivity[7]. This reducedtopology mustbekept updatedfrom time to timebecause thenetwork needsto evolveasthatof the active nodesare dying. Thepower controlon the radio transmitterhas notonlyan effect onthelife ofthebatteryof the nodes, butalsoonthecapacity of thetraffic. The control moduleof the power isoftenincorporatedin the protocolsofthenetworkoreitheroftheMAClayer. In [8], the authors demonstrate that there is an optimal scope of radio signal which minimizes the energy dissipated while maintaining connectivity. When the array of sensors is particularly dense, it is desirable to limit the closest neighborhood nodes to reduce collisions. in this optical, several algorithms topology control exist, especially DURING and DLMST that build the topology reduced based on the collected information locally.

With DLST [9] (Directed Local Spanning Subgraph), each node knows its position and that of its neighbors in one hop. Each node computes a minimum local spanning tree from its direct neighbors. The construction of the topology is based on the local node of each tree. with DRNG [9] (Directed Relative Neighbor- hood Grapheach sensor knows its position and distribute to its immediate neighborhood. With the position of its 1-hop neighbors, a node will deselect the longest links.

III.1.2. POWER MANAGEMENT

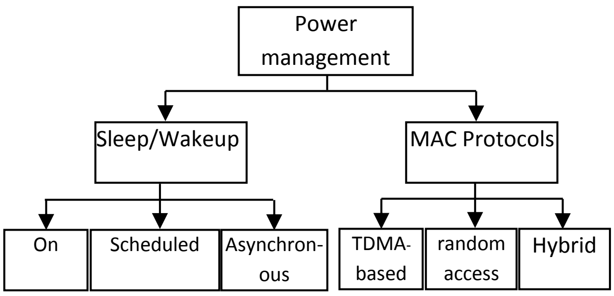


FIG.4.Classification of power management techniques

A. PROTOCOLES VEILLE-SOMMEIL (SLEEP/WAKEUP)

As mentioned earlier, a system sleep / wakeup can be defined for a given component (ie the Radio module) of sensor node. We can meet the principal planes sleep / wake up established as independent protocols above the MAC protocol (ie at the network layer and the application layer). Protocols sleep / wakeup are divided into three broad categories: on-demand, scheduled Rendezvous, asynchronous systems.

- ***Protocols on demand*** use the most intuitive approach to power management. The basic idea is that a node should wake up only when another node wants to communicate with him. The main problem associated with diets demand is how Tell a dormant node another node is willing to communicate with him. To this effet, these systems generally use multiple radios with different trade-offs between energy and performance (ie a radio at low rate and low power for the lights, and a radio "up" rate but in higher consumption for data communication). The protocol STEM (Sparse Topology and Energy Management) [10], for exemple, uses two radios ;
- Another solution is to use an approach to ***scheduled rendezvous***. The idea is that each node has to wake up at the same time as its neighbors. Typically, nodes wake up following a schedule of waking and remain active for a short period of time to communicate with their neighbors. Then they go back to sleep until the next *rendezvous*;
- Finally, a sleep/wakeup asynchronous protocol can be used. With asynchronous protocols, a node can wake up when they want and as long as he is able to communicate with its neighbors. This object is achieved by the properties involved in the sleep/wakeup scheme, no exchange of information is then required between nodes. Some diets sleep/wakeup asynchronous proposed in [11].

B. MAC PROTOCOLS WITH LOW DUTY CYCLE (MEDIUM ACCESS CONTROL)

Several MAC protocols for wireless sensor networks have been proposed, and many states of the art and introductions to the MAC protocols are available in the literature (eg [12]). We mainly focus on issues of power management rather than the channel access methods. Most of them are implementing a plan with a low duty-cycle to manage energy consumption. We have identified the most common MAC protocols by classifying into three categories : TDMA-based MAC protocols Contention-based protocols and hybrid protocols.

B.1. TDMA-BASED MAC PROTOCOLS (TIME DIVISION MULTIPLE ACCESS)

In theMACprotocol basedontime divisionmultiple access(TDMA) [13], each node hasa timewhere he canaccess the channeland usingall the bandwidthallocatedby the sensorfor transmission.TRAMA[14] is among thefirst techniques oftraffic management intheprotocolbased onTDMAnetwork. TRAMA, based on a distributed algorithm, provides a dynamic allocation of time slots within two hops. The access time to the channel is divided into two periods. In the first access period, the nodes exchange the addresses of neighbors for having a topology of all the neighbors located to two hops while in the second period the time is divided into several time intervals.

B.2. RANDOM ACCESS MAC PROTOCOLS

Random access means that all nodes share the transmission channel have the same right of access to it. This solution allows adaptability vis-à-vis the density and changes in network topology. However, it is also subject to access conflicts that lead to collisions and hence packet loss during transmission and energy at the MAC layer. CSMA and CSMA / CA. The multiple access with listening with carrier(CSMA for Carrier Sense Multiple Access) is introduced by (Kleinrock and Tobagi 1975).

In CSMA a node wishing to transmit a message, listen to the channel to determine if it is occupied. In this case, the node waits a random time before attempting retransmission. When the channelbecomes free, the node transmitsthe message immediately. An extended version of CSMA with Collision Avoidance CSMA / CA (CSMA / CA for CSMA Collision Avoidance) was proposed. She adds mechanisms to limit the number of message loss when neighboring nodes transmit at the same time. Wireless networks avoid collisions by exchanging control messages to reserve the channel before each data transmission.

The authorsin [15] proposed the MacprotocolS-MAC (SensorMAC). The main idea ofthis protocol is todivide the timeof the sensorinto two parts:activeandsleep. When the sensor gets activated, the sensor is ready to transmit or receive data while in the sleeping state, the sensor can neither transmit neither receive data. However, it is impossible to change the active and sleeping time after sensor deployment which prevents the sensors to adapt to different levels of traffic. It is for this reason that the authors in [16] proposed T-MAC (MAC Timeout).

In T-MAC, nodes are synchronized to get into active mode and sleeping periodically like S-MAC. However,thetime interval in T-MAC is not fixed by the application, but varies according to network traffic. If traffic is important, the sensors remain longer in active mode to transmit more data and if traffic is light, the sensors remain active shortest time to save energy. Alwaysbased on the methodofsettingactive mode andsleepperiodicallysensors, D-MAC [17] provides a verylow latencywhen comparedwith otherMACprotocolsin the samefamily. The purpose of D-MAC is that all nodes included in the multi-hop path to the sink are in active mode when data is being sent.

B.3. HYBRID MAC PROTOCOLS

A third family of protocols proposes to combine the two methods:

CSMA and TDMA. Thus, these protocols are trying to have the advantages of both methods by alternating the two in time or by combining them intelligently.

Z-MAC (Zebra-MAC) [18] dynamically changing transmission mode between CSMA and TDMA based on the current network load. Z-MAC uses CSMA as a basic protocol for media access but uses a TDMA scheduling to improve the resolution of contention between nodes. Crankshaft[19] is a hybridMACprotocol.In this protocol, time is divided into framesandeach frame is composedofseveralslots. To receive data, each node selects a unicast slot and all slots of broadcast during which he listens to the channel. Ainsi, thus Crankshaft, ordinance the data reception rather than sending. Funneling-MAC [20] is a hybrid MAC protocol that deals with the problem of congestion and packet loss observed by nodes near of the well. The protocol uses TDMA in this region and CSMA elsewhere. TDMA is managed by the well. All nodes use CSMA by default, unless they receive a message of the well informing them should use TDMA.

III.2. DATA-DRIVEN TECHNIQUE

the data-driven technique in wireless sensor networks aims to reduce the amount of data to be processed and transmitted. This technique can be classified according to the stages of data processing, into three categories: data acquisition, processing and transmission.

III.2.1. DATA ACQUISITION

Some physical magnitudes measured by the sensors do not change between two samples, it is the case for example where a temperature measurement où la dynamique est lente. This has encourage researchers to exploit the temporal correlation of the data. Techniques based on the collection of data can be classified into two categories: technique based on the prediction and techniques based on sampling.

– ***Technique based on a data forecast*** : A model of forecasting is established during the sampling data, so that future values can be predicted with some accuracy. This approach exploits the model obtained for reducing the number of acquired data, and also the amount of data to be transmitted to the well.

An example of the forecasting technique centralized been proposed in [21]. The authors proposeda prediction modelforenvironmentalmonitoring applicationsnamed "PREMON" which is based on twoprinciples.The firstis to exploitspatial correlations. Recognizing the efforts of a sensor, it becomes unnecessary for a neighbor to retransmit all packets received but only the relevant changes.

The second is based on the fact that an preview of network may be considered an image and therefore the evolution of the measurements can be viewed as a video.

By exploringthe concept ofMPEG,PREMONgenerates aprediction modelin the well andsentperiodically to thesensor nodes. Indeed, afterdeployment,thesensor nodessend theirinitial actionsat the well.

then the well then calculates the predictive model by exploiting the correlations between the data and sends it to the sensor nodes. Thus, upon receiving the model, the sensor node compares its data with the estimated value of the forecasting model. If the two values are close, the sensor node does not transmit its measurement at the well. (Vuran and Akyildiz, 2006) proposed a technique reducing the number of transmissions. The idea is to use the spatial correlation among a dense set of nodes to reduce the number of nodes relaying the event. It thus limits the number of collisions, improves transit time and saves energy. In [23], a prediction model based on clustering was proposed named ASAP. The authors of ASAP propose to regroup sensor nodes that have the same measures in the same cluster. The cluster-head and the well maintains the forecast model. And measures only above a certain threshold are transmitted to the well. The protocol "buddy" presented by [22], is based on the same principle as PREMON but using a distributed scheme to exploit the correlations between data. In the "buddy" protocol, sensor nodes are grouped together to form clusters and in each cluster, one node is chosen to represent the group.

– ***Technique based on the sampling*** : Techniques based on sampling can be classified into three categories: Adaptive sampling, hierarchical sampling and active sampling based on a model. As measured samples can be correlated, the adaptive sampling techniques exploit these similarities to reduce the number of acquisition.

The hierarchical sampling approach assumes that the nodes are equipped with sensors (or detectors) of various types. As each sensor is characterized by a given resolution and associated energy consumption, this technique dynamically selects the class to turn to obtain a compromise between precision and energy saving. The active sampling based on a model adopts a similar approach to forecasting data. A model of the phenomenon being measured is established during the sampling data, so that future values can be predicted with some accuracy. This approach exploits the resulting model to reduce the number of data samples, and also the amount of data to be transmitted to the well, although this is not their main goal.

III.2.2. PROCESSING AND DATA TRANSMISSION

Several approaches to data processing have been proposed in the literature to address the problem of energy conservation in wireless sensors. We present the main two. The first is based on data compression and the second on the aggregation of data.[24]

– ***Data Compression*** **:** The critical parameter networks of wireless sensors is their lifetime. But which are costly energy is communications. The advantage of the methods of compression is to allow less data to communicate, and thus to save energy, which results in an elongation of the life of the networks. However, compression algorithms commonly used on computer (such as Lempel-Ziv, WinZip or JPEG) are not directly transferable to all sensors, as many assume too many calculations in comparison to the gain provided. There are several methods of data compression. The main compression methods suited to wireless sensor networks data are:

– ***The coding by scheduling*****:** Compression in coding by scheduling is to remove redundant information from the sensors (the destination address, control code errors, clock synchronization) and merge the remaining data. Thus, the number of packets sent is reduced. In coding by scheduling, nodes send their packets to a single node called « node compression ». The encoded packet is constructed from combinations of the packets received from several nodes. The protocol "Funning" [25], based on the scheduling approach considerably reduces the amount of energy consumed during communication of up to 44% energy saving. This objective can be achieved by only sending a single data stream to a group of sensors instead of each individual sensor sends its data.

– ***the compression distributed***[28] **:** Each sensor provides a binary representation of its samples taking into account the correlation with the samples measured by the other sensors. Communication between sensors is forbidden which makes the estimation of this correlation difficult in practice.

– ***data agrégation*** : Sensor networks are dense enough in general, this means that nodes close enough (neighbors) can capture the same data (temperature, pressure, moisture equivalent, for example). Therefore, several studies have been conducted to eliminate redundancy and reduce data traffic in the network. The mechanism consists in treating the data collected by each sensor in a node called « aggregator node ». Only the result produced will be transmitted to the base station. In this way, the amount of data communicated in the network can be reduced, which consequently reduces the consumption of bandwidth and depletion of sensors energy.

The data aggregation techniques can be divided into two:

– ***centralized aggregation*****:** Aggregation in the clusters, formed via a clustering protocol. First zones (clusters) are defined via a protocol then the data are aggregated in these areas through a cluster-head. the latter may may possibly change over time in order to better distribute energy consumption among all nodes in the network.

– ***distributed aggregation*** **:** Aggregation is distributed in an aggregation tree that is to say that the network is viewed as a whole. COUGARprotocol[26] is an example of aggregationofdistributed data. The data producedby the networkare modeled asa relational table,the network isseen asa largedistributed database.The attributesof thistableare eithersensor informationor dataproduced bythe sensorbecauseCOUGARprovides partialaggregationat nodes.

III.3. MOBILITY TECHNIQUE

In some cases where nodes are mobile, mobility can be used as a tool for reducing energy consumption (beyond the duty-cycling and data-driven techniques). In astaticnetworkofsensors, packets fromnodesfollowmultihoppaths tothe base station. Thus, some paths can be loaded (solicited more than others), and the nodes nearby to the base station relaying more the packets and are more prone to premature depletion of their batteries (funneling effect). If some nodes (possibly the base station) are mobile, traffic can be changed if the mobile nodes are responsible for collecting data directly from static nodes. Ordinary nodes are waiting for the passage of a mobile device to send their messages so that communication takes place nearby (directly or at most a limited number of hops). Therefore, ordinary nodes can save energy because the path length, contention and overheads of diffusion are thus reduced. In addition, the mobile device can visit the network to evenly distribute uniformly the energy consumption due to communication. When the cost of the mobility of sensor nodes is prohibitive, the traditional approach is to attach a sensor to entities that are roaming in the sensing field, such as buses or animals. Strategies based on mobility can be classified into two groups: strategies with a mobile Sink and strategies with mobile relays, depending on the type of the mobile entity. It is important to note here that when we examine mobile systems, an important problem is the type of control the mobility of nodes that Integra network design, this is detailed in [27]. Mobile nodes can be divided into two categories: they can be specifically designed as part of the network infrastructure, or part of the environment. When theyare part of theinfrastructure, mobility can be fullycontrolledto the extent thattheyare generallyrobotic. Whenmobile nodesare part ofthe environment, theymay notbecontrollable. If they follow a strict schedule, they have a completely predictable mobility (eg, a shuttle for public transport). Otherwise, they may have a randombehaviorsuch that noassumptioncanbe madeontheirmobility. Finally, they can follow a pattern of movement, which is neither predictable nor completely random. For example, in the case of a bus moving in a city whose rate is subject to significant variation due to traffic conditions. In this case, the mobility patterns can be drawn based on observations and estimates of some precision.

VI. CONCLUSION

In this article, we presented a state of the art works found in the literature, which deal mainly improving the lifetime of a network of wireless sensors. We presented first in a brief manner the different sources of energy recovery in the external environment such as solar energy, electromagnetic waves and thermal energy. In the remainder of this paper we describe the different approaches to existing energy conservation used to extend the lifetime of a network of wireless sensors. To this end, several methods have been developed either as MAC protocols, low duty-cycle or as independent higher level protocols based on scheduling sleep / wakeup and there are other methods that is interested in data acquisition and processing and transmission of data and methods that use the technique of node mobility. There are of course many other techniques of energy conservation. For example, from paradigms of self-organizing systems, cross-layers and other mechanisms independent network protocol level or application level.

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Incremental Design of Nano-Sensors Network

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Abstract

A general approach is to reduce the overall complexity by raising the level of abstraction of the design phase. On the context of this idea, we suggest to apply incremental formal approach in the context of nano-devices network and wireless body network(WBAN).The suggested research approach consists of constructing architectures based onNoC-FPGA(Network-on-Chip-field Programmable Gate Assembly). This research work includes the mapping of this kind of architecture on some special context likeWirelessnano-sensors network.The methodology used the formal method based on validated B-event theories to generate the chosen context.The used theories are NocTheory, wireless network theory,colored graph theory and theVHDL (Very High Speed Integrated Circuit Hardware Description Language) theory.Properties like reliability, fault tolerance will be crossed on the kind wireless nano-sensors network.

Key-words: Nano-sensor network, NoC, Formal methods, B-Event, Theory Concept, FPGA, VHDL, WBAN

1. Introduction

Several research works explore systems verification using formal technics in computer science but not all these works perform general methodology which integrates design flow of micro-electronic systems. This validation is more complex in the case of strong reconfigurable communicating architectures [1,14] using Nano metric scale [5,10,13] which don't consume energy, without interference and ultra wide band (UWB) based on impulsion [2,3,11,15,17]. Mac layers differs from traditional one [10] on which fair energy consuming, good synchronization precision and compatible with nanotechnologies. The liability and security of this kind of wireless sensors become important research study. [16]

The main requirements in the domain WBAN are low power consumption, low latency and high reliability communications [11,12]. The deployment of devices based on nanomaterials inside the human body correspond to various physiological parameters for example to monitor on a real-time manner the imbalance of cholesterol, the measurement of bone growth, etc

The most important characteristic in the WBANs which is variable over time is the stored energy of the nanomaterials. During communication cycles, the energy is dissipated and can be reconstituted using certain energy recovery techniques (rhythmic vibration of the heart, body movement, ..), resulting in an energy fluctuation node aware of the time. Therefore, the lifetime of this type of network is longer.

Our suggestion consists of suggesting a new approach of the design of sensors network using fault-tolerant platform [6,7] based on FPGA reconfigurable technology. This approach uses proved incremental refinement notion in order to improve validation phase (by reducing time and reaching certain reliability) of high level design. The purpose of the study is to suggest using an incremental manner a new flow design which will generate synthesizable architecture of fault tolerant sensors network using FPGA technology. This purpose will be possible by investigating formal proof using Rodin Tool associated to Event-B Language [8]. The micro-architecture will be composed by two parts: the first part consists of the nano-device [10] and the second part consists of the device constituted by SoC using FPGA technology [14]. This platform allows us to design and simulate wireless sensors using IPs (intellectual property) in order to develop advanced embedded applications like for example WBAN and nano sensor network

The important part of the study consists of representing the network using several theories like graph theory [6], colored graph [20], VHDL theory [21]. Event- B language helps us to represent this mapping. The mechanism of fault-tolerance will have enriched using the notion of refinement. [18]

The paper structure is as follows: section two presents base Event-B theories. This section shows how we can model self-organized network using colored graph theory, VHDL is also presented. Section three develops a part of incremental Design of Nano-FPGA architecture deployment using BAN context. We conclude with the main ideas which allow us to combine two parts: analogic (nano-sensor) and digital parts (networkof FPGAnodes) in order to prove the properties of this kind network.

2. Generic Event-B models

Models in Event-B are specified by means of contexts (static properties of a model) and machines (dynamic properties of a model) and during the modeling of every system, the main benefit in this work is the use of models from theories already validated to reduce the time in the formal specification of any given system taking as target the NoC system after n-level of verified models, a theory (a new kind of Event-B component) is defined independently of any particular models. A theory component (Figure 1) has a name, a list of global type parameters (global to the theory), and an arbitrary number of definitions and rules: The Theory plug-in provides a mechanism to extend the Event-B mathematical language as well as the prover. The main purpose of this new feature was to validate systems with a way to extend the standard Event-B mathematical language by supporting operators, basic predicates and algebraic types. Along with these additional notations, it can also define new proof rules (proof extensions). A theory is used to hold mathematical extensions: data types, direct, recursive and axiomatic definitions of operators, and proof extensions polymorphic theorems, rewrite and inference rules. Theories are a helpful basis for the static checking and proof obligation generation which ensure that no compromise for the existing infrastructure of modeling, proof and any contributed extensions. In essence, the theory plug-in provides a systematic platform for defining, validating of the well-creation of embedded systems.

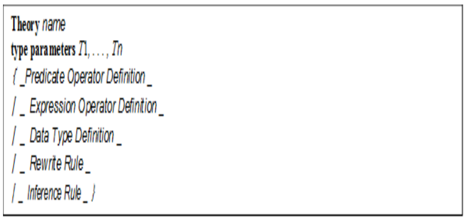

Theory name
type parameters T1, ..., Tn
{ _Predicate Operator Definition_
| _ Expression Operator Definition_
| _ Data Type Definition _
| _ Rewrite Rule _
| _ Inference Rule _ }

Figure 1.Theory inEvent-B

This work proposes to validate very particular the wireless sensors network based on BAN systems (Figure.2.) which is composed of set of NoC switches using the NoC theory with the graph theory, the new strategy for recovering the set of faulty nodes in form of theory(WSNoC theory) to handle and manage the complexity of the multi-failure of sensors using the colored graph theory, the final step is to ensure that all the properties for this embedded system will applied correctly in the application environment by combining these previous theories with the VHDL. The last step will allow us to launch a set of scenarios to check all possible case for running correctly the BAN system.

In this paper we present some important points:

The benefit of using generic theories to model a BAN-Based wireless sensors network in incremental fashion to ensure the most valuable properties used in this kind of network.

The token case study is for analyze the reasons of collision during multiple sending of data and propose a mechanism to detect the failure and solved it using a set of theories that help in the final step to generate a VHDL code thanks to VHDL theory.

The carry out of some temporal properties in the modeling phase to make the proposed recovering strategy more convenient.

A quick view of the different validation steps of the BAN system designing, starting from the conception using Event-B generic theories that help to implement with vivado Environment to simulate and upload the perfect VHDL codes to the FPGA board.

2.1. NoC-based Wireless Network theory modeling using the Event-B

The creation of a very particular NoC system inspired from the model proposed [7] for a self-reconfigurable multi-node network, this net is composed of a set of self-organized wireless nodes. Each node is independent. This allows maintenance andoperational reliability of the system in the case of failure.

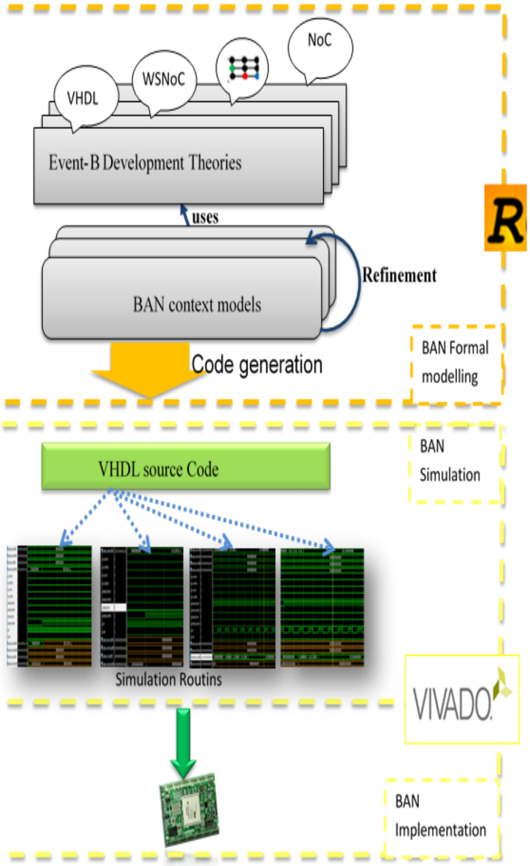


Figure 2. BAN system designing flow

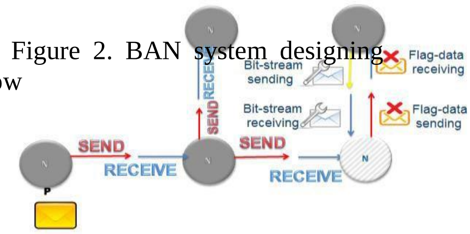


Figure 3. The Wireless network of NoC theory in Event-B

This network system is a system composed of several nodes which apply XY algorithm inside as described previously and communicate with each other through a communication channel, which can be a network, shared variables, messages, this allows us to consider our system like a distributed system which is often seen as a graph [20], where the vertices are the nodes and the edges, direct communication links between them.

During the modeling for this particular network (Figure 3.) used NoC, it must introduced the way of reconfiguration for any faulty node and for this new constraint it adds a new extension for the NoC theory, this new theory Wireless-NoC contains:

The new data-type “state” is used to represent the state of every node.

The next operators are used to clarify more the new theory that respects all the properties of reliabilities: WNocStat: to create a set of nodes with its state:

ocpy: mentions that the node is occupied by sending or receiving data. free: mentions that node is not occupy.

nodeState: this operator presents the state of specific node "nod".

packet: to represent the new structure of data which contains a flag "flg" to specify the type of data in the

case of node failure this flg will have the value 1.

Datatype *state* **constructors** *ocpy* *free*	**operator** WSnode **prefix** **args** s: state , nod: $\mathbb{P}(S)$ **condition** $any \in role$ $\wedge\ net \in node(any, nod)$ **definition** net	**operator** packet **prefix** **args** $flg: \mathbb{N}$ data: $\mathbb{P}(T)$ **condition** $flg \in 0..1$ **definition** data	**Operator** nodeState **prefix** **args** nod: $\mathbb{P}(S)$ **condition** $a \in role$ $\wedge\ net \in node(a, nod)$ $\wedge\ s \in state$ **definition** s

So it can present this particular NoC using Wirless-NoC theory definition like in follow:

Data sending

When a source Srcnod sends a packet (msg), the packet must have a value of 0 for the flg argument and the states of nodes Srcnod, Desnod must be changed with the value ocpy.

Data recieving

A packet is received by its destination, if the packet has reached the destination (node with a role equal to rcv Case 3 in predicate).This packet must have a value of 0 for the flg argument and the states of nodes Srcnod, Desnod must be changed with the value free.

Data forward

In the network, a packet (msg) transits from a node (Srcnod) to another node Desnod(node with a role equal to dst Case 2 in predicate).This packet must have a value of 0 for the flg argument and the state of the node Srcnod must be changed with the value free when the node Desnod will have the value ocpy.

Case 1: SEND
$Send = send \cup \{Srcnod \mapsto msg\} \wedge srcstate = ocpy \wedge Desstate = ocpy$
Case 2: FORWARD
$rcvd = rcvd \cup \{Desnod \mapsto msg\} \wedge srcstate = free \wedge Desstate = ocpy$
Case 3: RECIEVE
$rcvd = rcvd \cup \{Desnod \mapsto msg\} \wedge srcstate = free \wedge Desstate = free$

In the optimal case those last events could represent the communication between different nodes otherwise some nodes could be in failure state so they must inform the other nodes by sending a flagData with flg=1, (see Flag data sending).

Flag-data sending

In case of a node (Fnod) Failure which can't receive a packet (msg), the packet FlagData with a value of 1 for the flg argument is sent if the faulty node Fnode still be occupied (value ocpy).When the flag data is received (see Flag-data receiving), one of the healthy nodes take the mission of reconfiguration (see Config-data receiving) for the faulty nodes after checking these following rules:Every node is not a faulty node, has the material resources to implement IP node, has the Bitstream packet of configuration IP and do not be occupied by a priority.

$\forall send, Fnod, FlagData \cdot FlagData \in packet(1, \mathbb{P}(T)) \wedge srcstate = Statenod(Srcnod)$
$\wedge\ failstate = Statenode(Fnod))$
$\Rightarrow$
$Send = send \cup \{Fnod \mapsto FlagData\} \wedge srcstate = ocpy \wedge Failstate = ocpy$

Flag-data recieving

A packet (FlagData) is received by switch (Srcnod), if it is sending from a faulty node (Fnod).

$rcvd = rcvd \cup \{Srcnod \mapsto FlagData\} \wedge srcstate = ocpy \wedge Failstate = ocpy$

Config-data sending

In the case of a node (Srcnod) can check the rules of reconfiguration ability (described before)it send a configuration data (Bitstream) to the faulty node (Fnod).

$\forall Bitstream, FlagData \cdot Bitsream \in packet(0, \mathbb{P}(T)) \wedge FlagData \in packet(1, \mathbb{P}(T))$
$\Rightarrow Send = send \cup \{Srcnod \mapsto Bitstream\} \wedge srcstate = free \wedge Failstate = ocpy$

Config-data recieving

A packet (FlagData) is received by switch (Srcnod), if it is sending from a faulty node (Fnod).

$rcvd = rcvd \setminus \{Fnod \mapsto Bitstream\} \wedge srcstate = free \wedge srcstate = free$

2.2. Vertex coloring theory modeling

In the reason of managing nodes failures and the way of the self-recovering, the graph theory helps to specify the Wireless reconfigurable WS-NoC(Figure 4.) using algorithms of colored graph principals [20] that include a part from Closure theory, the closure theory as in follow composed of operator cls and have so many properties of graph such as composition of sub-closures an transitivity

THEORY closure **OPERATORS** •**cls** : $cls(r : \mathbb{P}(S \times S))$ **direct definition** $cls(r : \mathbb{P}(S \times S)) \triangleq fix(\lambda s \cdot s \in \mathbb{P}(S \times S) \mid r \cup (s;r))$ **THEOREMS** $\forall r \cdot r \in \mathbb{P}(S \times S) \Rightarrow cls(r) = r \cup (cls(r);r)$ $\forall r \cdot r \in \mathbb{P}(S \times S) \Rightarrow r \subseteq cls(r)$ $\forall r \cdot r \in \mathbb{P}(S \times S) \Rightarrow cls(r);r \subseteq cls(r)$ $\forall r, x \cdot r \in \mathbb{P}(S \times S) \wedge r[x] \subseteq x \Rightarrow cls(r)[x] \subseteq x$, $\forall r, s \cdot r \in \mathbb{P}(S \times S) \wedge s \in \mathbb{P}(S \times S) \wedge r \subseteq s \wedge s;r \subseteq s \Rightarrow cls(r) \subseteq s$ $\forall r \cdot r \in \mathbb{P}(S \times S) \Rightarrow cls(r);cls(r) \subseteq cls(r)$		$\forall clr, count \cdot clr \in COLORS \wedge Count \in \mathbb{N}$ **Case 1 :colored_faulty** $\wedge count \leq 4$ **Case 2 :faulty_wait_to_be_colored** $\wedge count > 4$ **Case 1 :colored_faulty** *colored*= colored ∪ {Fnod} ∧ colored(Desnod)=clr ∧ count=count+1 ∧ has_colored=has_colored ∪ {Fnod} **Case 2 :faulty_wait_to_be_colored** *wait_to_be_colored*= wait_to_be_colored ∪ {Fnod} ∧ count=count+1 ∧ colored(Desnod)=clr
Datatype *Colors* **constructors** *Yellow,Red,* *Green,Bleu ,none*	**Operator** Colorenode **prefix** **args** c:Colors , nod: $\mathbb{P}(S)$ **definition** nod	

The coloring graph algorithms [20] can be used to control a set of nodes: There may be two nodes that have the same job but two adjacent nodes cannot even fix the failed node at the same time [17].

Math extension is a standard library provides the closure as a theory which is almost similar to our Wireless NoC architecture but need to add the coloring rules in the context to cover all courant proprieties of our Graph Theory.

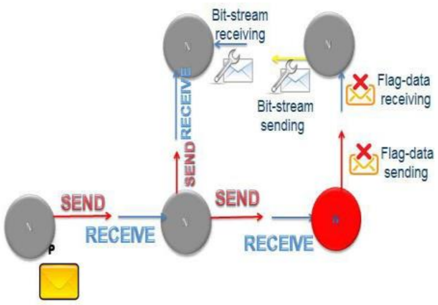


Figure 4. The colored graph theory application onto WSNoC system in Event-B

During the refinement events which are forthe

Application variables represented by a graph theory and their Execution environment variables must be handled in the same model by introducing the VHDL theory (see section 2.3) that could be used in every event Model as new VHDL variables represented with Event-B. So the coloration will cover all possible state for the events that nodes can do.

So any node that cannot send and receive data can be labelled with Blue ,Yellow, Green , Red and uncoloured even managed the multi-failure of nodes that overpass more than 4 nodes by counting it as node in waiting list to be in the next colored due to the available colour.

Flag-data sending

In case of a node (Fnod) Failure which can't receive a packet it must colore the node that sends a

FlagData with red.

```
Send=send∪{Fnod↦FlagData} ∧colored(Fnod)=clr ∧has_colored:=has_colored∪{Fnod }
```

Config-data sending

Bitstream-packet (Bitstream) is received by a faulty switch (Fnod), if it is sending from a reorganizer IP (Rfcgnod) in the buffer, then for that reason both nodes Fnod and Rcfgnod must be uncolored.

```
rcvd = rcvd \ {Fnod↦Bitstream }∧colored(Fnod)=none ∧has_colored:=has_colored\{Fnod }
```

2.3. The VHDL theory in Event-B

The Theory VHDL_th (Figure 5) used during this Self-Organized network Modelling is composed of an Entity and a set of architectures that contains a set of variables, the entity contains ports in direction in out or in_out, from that two operators are created (arch_decl and ports_decl) when one had a data type parameter (pio), in this VHDL case study, we had std_logic and the std_logic_vectors signals, the last one must have new theory could be used in the VHDL_th, it is a similar structure of array presented in[34].The variable in VHDL must have a type and in this structure for example the integer type is represented like an operator and ensure that couldn't be over int_max= and every operation that use that kind of variable must never overflow that value so the Proof obligation will ensure the non-overflow error for integer variables in the VHDL code. During the modelling of our VHDL theory that it will introduce into the Event-B model the set of operations created in our new theories and taking some cases like VHDL operations (+,*,-, /), or the assignment of variables in VHDL, also some proofs had to be discharged during this WSNoC modelling will be a good start for the code generation after ensuring the well definition of all the system.

```
THEORY VHDL_th
DATATYPES
pio≙I n , out, inout
OPERATORS
•arch_decl :
arch_decl(s:ℙ(T))
direct definition
arch_decl(s : ℙ(T)) ≙ s
•port_decl : port_decl(p:pio,s: ℙ(T))
direct definition port_decl(p:pio,s : ℙ(T)) ≙ s
•std_logic : std_logic
direct definition
std_logicl ≙ 0‥1

•vector:
vector(s: ℙ(T))
direct definition
vector(s : ℙ(T)) ≙
{n,f·n∈ℕ∧f∈0‥(n−1)→s|f}

•std_logic_vector:
vector(length: ℕ,s: ℙ(T))
well-definedness condition
direct definition
std_logic_vector(length: ℕ,s : ℙ(T)) ≙
 {v|v∈vector(s)∧card(s)=length}
```

Figure 5. The VHDL theory

3. Theories deployment and WBANContext

To check the validity of the network designed in the context of WBAN, we rely on three theories: colored graph theory, theory NoC, the theory WSNoC. The wireless network BAN context type is based on the probabilistic model [10], which we have extracted several parameters such as packet rates, time of energy consumption, ... These parameters are injected into the machine -Event B BAN (see figure 6.a,6.b.and 6.c.). The context of WBANs includes several types of collisions these collisions occur when a nano- device initiates its transmission in the interval when the noise of the molecular absorption is at its peak, or the receiver's power is low. Nano-devices are unavailable for certain duration. The two factors responsible for this type of collision are the behaviour of the channel, and the fluctuation of the receiver's power. The generic model presented in the previous sections incorporates nodes status. In this refining step we integrate the channel state when the transmitted packet does not undergo significant molecular noise absorption, and this packet can be detected at the receiver with a highest probability; by cons put it there's case the packet is lost due to the significant molecular noise absorption.

On this modelling we try to evaluate the state of receiver node during a multi-transmission of data using the following invariants:

p: The number of receiving data during a single emission

q: the non-received data number during a single emission

the node will take the state of a faulty node if the number of received data still less than the total data and the duration of the emission overpass the threshold of an ordinary emission task (model Ban_M0), though the faulty node will be colored using Colored graph theory(Ban_M1).

when the study that we have is about a multi nano-devices that send multiple symbols, so the failure of several nodes lead to the failure of the system, this is why we need to colored all faulty node till the recovery of the nodes by sending the bitstream data which is considered as the main properties covered by the WNoC theory, so every recovered node will be uncolored till the end of all the faulty nodes which are the waiting list(Ban_M2).

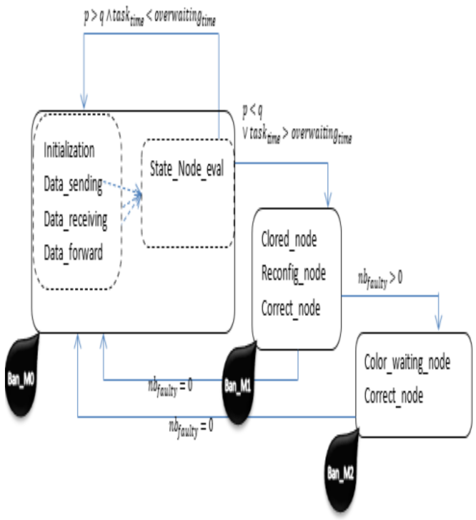
The flow of data must not be randomly, but it must follow four case to check the behavior of nodes respecting the WSNoC theory rules, NoC theory and especially the vertex colored graph theory, we can clearly observe this behavior using ProB Tool delivered in the Rodin toolset as a scenario of animation where multiple data must be received by a node:

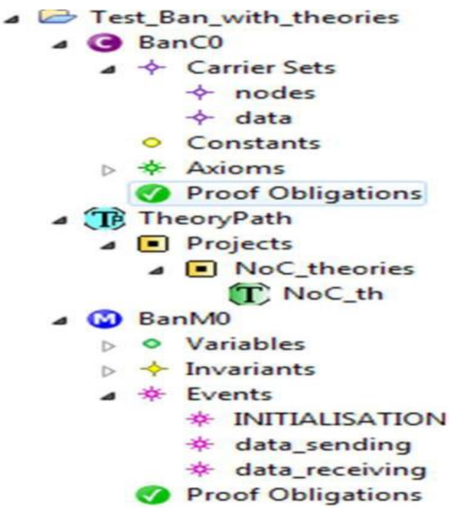
In the same time with a same rate (the arrival time to add for the Task_time will be the same even the time reserved for saving the data from a direction dir_buff_time).

In the same time with different rates (the arrival time to add for the Task_time will be the same and we will add different times reserved for saving the data from a direction dir_buff_time).

At different times with the same rate (the arrival time to add for the Task_time will be not the same but the time reserved for saving the data from a direction dir_buff_time will be the same).

At different times with different rates (the arrival time to add for the Task_time will be not the same even the time reserved for saving the data from a direction dir_buff_time).


Initialization
Data_sending
Data_receiving
Data_forward
State_Node_eval
Clored_node
Reconfig_node
Correct_node
Color_waiting_node
Correct_node
Ban_M0
Ban_M1
Ban_M2


Test_Ban_with_theories
BanC0
Carrier Sets
nodes
data
Constants
Axioms
Proof Obligations
TheoryPath
Projects
NoC_theories
NoC_th
BanM0
Variables
Invariants
Events
INITIALISATION
data_sending
data_receiving
Proof Obligations

```
MACHINE
      dftM0 (c)
SEES
      dftC0
VARIABLES
      sent            //  p envoyé
      rcvd            //  p reçu
      switchcontrol
      str
      gr
      Task_rate
      Emis_time
      Total_data
      Forward_time
      Rcvd_data
      Recp_task
      syst_state
EVENTS
      INITIALISATION  ≙
      STATUS
        ordinary
BEGIN
        act1   :  sent:=∅
        act2   :  rcvd:=∅
        act3   :  switchcontrol:=∅
        act4   :  str:=∅
        act5   :  gr:= g
        act6   :  Task_rate:=0
        act7   :  Emis_time:=2
        act8   :  Forward_time:= 3
        act9   :  Total_data:=0
        act10  :  Rcvd_data:=0
        act11  :  Recp_task:=∅
        act12  :  syst_state:=0
END

      SEND  ≙              //  l'envoi du p(sending data)
      STATUS
        ordinary
ANY
        p          //  paquet
        s          //  un noeud source
WHERE
        grd1   :  p∈PACKETS
        grd2   :  s∈NODES
        grd3   :  s=src(p)
THEN
        act1   :  sent:=sent∪{s↦p}
        act2   :  str:=str ∪{s↦p}         le réseau
        act3   :  Task_rate:=Task_rate+Emis_time
        act4   :  Total_data:=Total_data+1
END
END

      RECIEVE  ≙              //  la reception du p
      STATUS
        ordinary
ANY
        p
        d          //  dst
        s
WHERE
        grd1   :  p∈PACKETS
        grd2   :  d∈NODES
        grd3   :  s∈NODES
        grd4   :  s=src(p)
        grd5   :  d=dst(p)
        grd6   :  p∈ sent[{s}]
THEN
        act1   :  rcvd := rcvd ∪{d↦p}
        act2   :  str:=str\{d↦p}
        act3   :  Recp_task:=Recp_task∪{d↦Task_rate}
        act4   :  Rcvd_data:=Rcvd_data+1
END

      FORWARD  ≙
      STATUS
        ordinary
ANY
        n
        d
        m
        p
WHERE
        grd1   :  n∈NODES
        grd2   :  d∈NODES
        grd3   :  m∈NODES
        grd4   :  p∈PACKETS
        grd5   :  d=dst(p)
        grd6   :  n ↦ p ∈ switchcontrol
        grd7   :  n ↦m  ∈ cls(gr)
        grd8   :  m ↦ d ∈ cls(gr)
        grd9   :  str=str ∪{n↦p}          //  le p est mis ds le reaseau
THEN
        act1   :  switchcontrol:=(switchcontrol\{n↦p})∪{m↦p}
        act2   :  Task_rate:=Task_rate+Forward_time
        act3   :  Rcvd_data:=Rcvd_data+1
END
      EVALUATE1  ≙
      STATUS
        ordinary
ANY
        p
        p2
WHERE
        grd1   :  p∈ℕ
        grd2   :  p2∈ℕ
        grd3   :  p2=Rcvd_data
        grd4   :  p=Total_data
        grd5   :  p−p2≥Total_data÷2
THEN
        act1   :  syst_state:=1
END
END
```

Figure 6. the BAN context modeling.
a. the modeling flow of BAN models.
b. the first BAN model(Ban_M0) and the used NoC theory
c. The BAN Machine Ban_M0

4. Experimental test and results

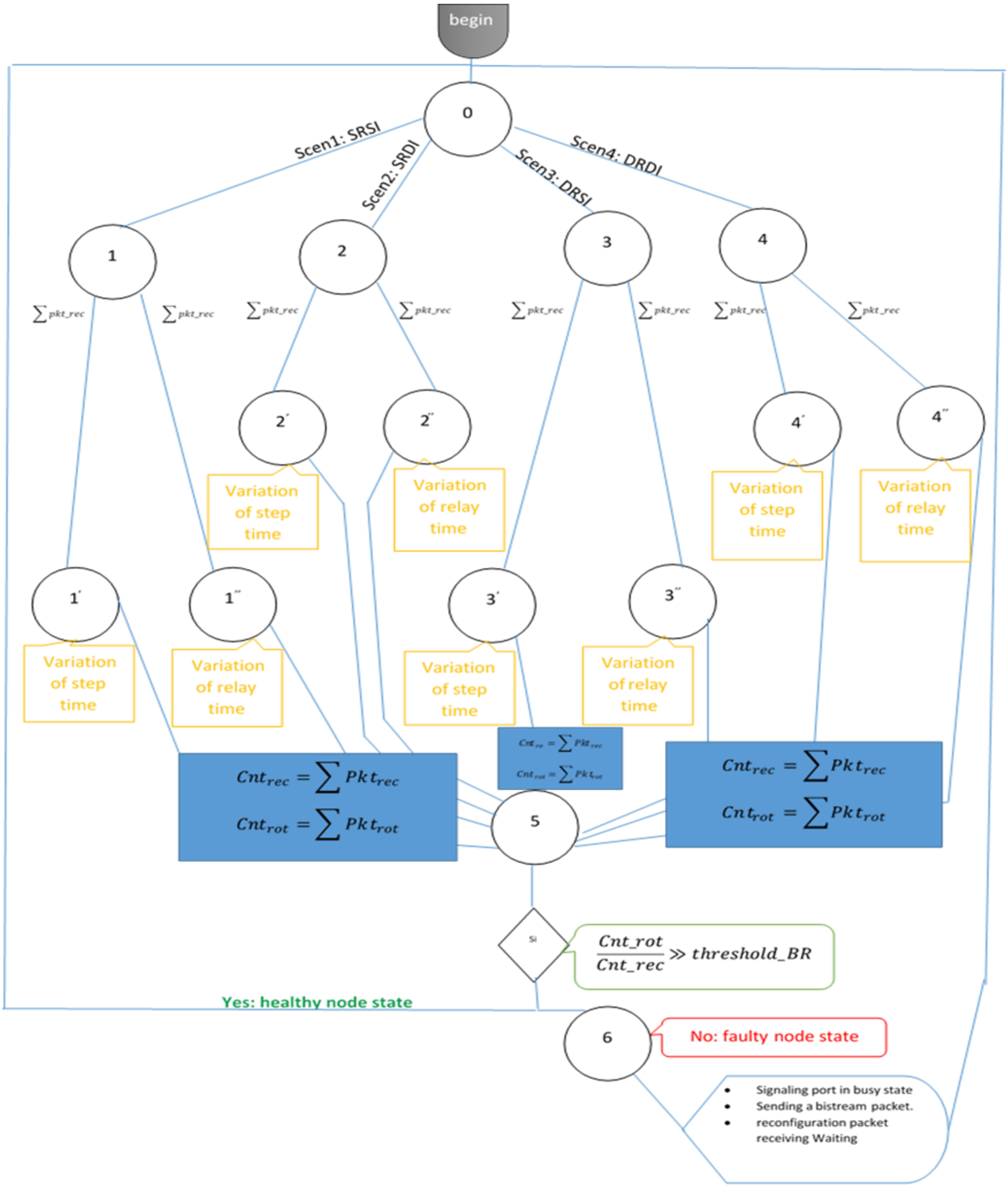


Figure 6 the simulation of BAN context on XilinxvivadoEnvironment.

It is important to coordinate concurrent transmissions from multiple nano-machines to realize the potential of multi-hop BAN network in the aim of answering this meaningful question: Does the process of sending different symbol rates by different bio-nano-devices invite any collision at the receiver?

We redefine “collision” in the context of BAN

networks. It includes not only sequential collision of symbols of a same packet but also the sequential collision of symbols from different packets.

We observe that two important factors are responsible for such type of collision: the instance for arrival of data, and the rate to make it received by a node.

Following the same line of argument just presented, we consider that the state of receiver node can be

Modeled using Event-B model that call the NoC theory, colored graph theory and VHDL theory, this model will also give the decision of how to give a recover signal to make the node in the state “faulty” become again a “healthy” node.

This model was interpreted to VHDL annotation thanks to the VHDL theory giving us the opportunity to created four axes of simulation where the sent data were received at:

same rate (speed associated to distance which corresponds in our case to direction) and same time instant. (or shortly SRSI see Figure 6).

same rate but different at time instants, (or shortly SRDI see Figure 6).

different rates at same time instant, (or shortly DRSI see Figure 6).

different rates and different time instant, (or shortly DRDI see Figure 6).

The variation of the two factors was to ensure the reason of collision as follow:

4.1. The Variation by Arrival Rate

The variation of the two factors was to ensure the reason of collision. Knowing that changing the rate of arrival data is to change the time scale that represent this reservation of time of data to ensure the passage from the Net interfaces to the buffers then to the next interfaces calculated.

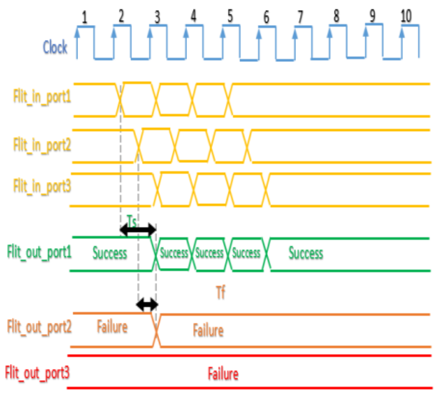


Figure 7 the rate variation meaning in BAN context

4.2. The Variation by Arrival Instance

The variation of the arrival instance was calculated by observing the escape time between the time of successful receiving data (Ts) in the net interfaces and the failure time (Tf).

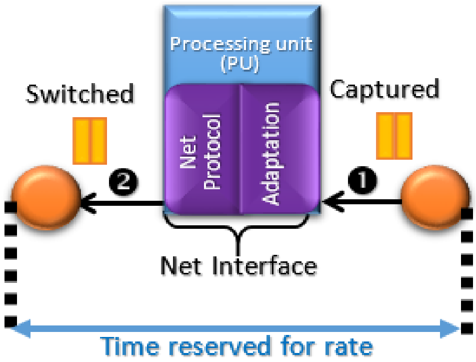


Figure 8 the time variation meaning in BAN context

5. Conclusion and future perspectives

This suggested new approach for fault-tolerant wireless network using FPGA base SoC (System-On-Chip). This approach exploits the notion of refinement and incrementality mechanisms and formal proof. It allows us to improve the reliability and decrease the design time of high level specification.

The strategy used during the study, well-defined graph theory, auto-self organization network using reconfigurable FPGA and well-defined nano-sensors theory. We can map theses nano-sensors theories with IP blocs. We show that our methodology can integrate several constraints like for example those associated to probabilistic model. Other concepts and technics associated to BAN network can be injected in future research such Multiple Input Output Orthogonal Frequency Demodulation Modulation MIMO-OFDM [11], as new step in the refinement process.

The methodology used the coupled tools, at one hand RODIN tool based Event-B language and at the other hand Simulation Xilinx environment [9] and VHDL language. Code generation and integration system will be performed in the last step of the methodology.

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