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# Invention and application of epoch of Kifilideen's Sum Formula for Kifilideen's General Matrix Progression Series of infinite terms in solving real-world problems

Kifilideen Lekan Osanyinpeju✉

Agricultural and Bio-Resources Engineering Department (ABE), College of Engineering (COLENG), Bells University of Technology, Ota, Ogun State, Nigeria

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**Abstract.** Kifilideen's general matrix progression series of infinite terms is the summation of values of a collection of progressive members of the numbers series system. The number series system or set has endless terms where terms are progressive into levels and steps (within level) with increasing members set in successive levels and just one member in the first level. This kind of series is needed in determining the overall value(s) of the collection in the progressive members of the system which is/are useful for budgeting, analyzing, accounting, allocating and planning the system of arrangement that adopts such series. This study invented and applied the epoch of Kifilideen's Sum Formula for Kifilideen's General Matrix Progression Series of infinite terms in solving real-world problems. The mathematical induction of sum formulas of bi–numbers product progression series was formulated and established. These sum formulas obtained were incorporated into inventing Kifilideen's Sum Formula for Kifilideen's General Matrix Progression Series of infinite terms. The Kifilideen's Sum Formula invented in this paper and Kifilideen's Components Formulas of the Kifilideen's General Matrix Progression Sequence of infinite terms were used in conjunction to proffer solutions to real-world problems. The established Kifilideen's Sum Formula for Kifilideen's General Matrix Progression Series of infinite terms provides an easy and fastening process of finding the summation and evaluation of the overall value(s) of collection of progressive members of the Kifilideen's General Matrix Progression Series of infinite terms.

**Keywords:** Sum Formulas, Matrix Progression Series, Kifilideen's Matrix Structural Framework, Matrix of Series and sequence, Matrix of Arithmetic Progression Series, Bi–numbers product progression series.

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✉ Corresponding author. Email: amkifilideenosanyinpeju@gmail.com , kosanyinpeju@bellsuniversity.edu.ng, prof_4us@yahoo.com

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# 1 Introduction

Kifilideen's General Matrix Progression Sequence of infinite terms is a sequence of patterns in which its members are arranged into levels and steps within the levels in Kifilideen's Matrix Structural Framework [9,12]. This kind of sequence has infinite terms having endless members [10]. In Kifilideen's Matrix Structural Framework for Kifilideen's General Matrix Progression Sequence of infinite terms, the $1^{st}$level has one step, $2^{nd}$ level has two steps, $3^{rd}$ level has three steps and so on. This type of sequence has endless levels in Kifilideen's Matrix Structural Framework. The Kifilideen's Structural Framework of the Kifilideen's General Matrix Progression Sequence of infinite terms is provided in Table 1.1.

**Table 1.1:** Kifilideen's Structural Framework of the Kifilideen's General Matrix Progression Sequence of infinite terms

| | | Level, $l$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | $l_1$ | $l_2$ | $l_3$ | $l_4$ | ... | $l_l$ |
| | $s_1$ | $k(0)+i(0)+f$ | | | | | |
| | $s_2$ | | $k(1)+i(0)+f$ | | | | |
| | $s_3$ | | $k(1)+i(1)+f$ | $k(2)+i(0)+f$ | | | |
| | $s_4$ | | | $k(2)+i(1)+f$ | $k(3)+i(0)+f$ | | |
| Step, $s$ | $s_5$ | | | $k(2)+i(2)+f$ | $k(3)+i(1)+f$ | . | |
| | $s_6$ | | | | $k(3)+i(2)+f$ | .. | |
| | $s_7$ | | | | $k(3)+i(3)+f$ | ... | ... |
| | $s_8$ | | | | | | ... |
| | $s_9$ | | | | | | ... |

For the sequence of the Kifilideen's General Matrix Progression of infinite terms is presented as:
$k(0)+i(0)+f;\, k(1)+i(0)+f,\, k(1)+i(1)+f;\; k(2)+i(0)+f,\; k(2)+i(1)+f,\, k(2)+i(2)+f;\, k(3)+i(0)+f,\; k(3)+i(1)+f\,,\; k(3)+i(2)+f\,,\, k(3)+i(3)+f;\, \dots, \dots, \dots,$
More so, According to [12], the Kifilideen's General Term Formula for the Kifilideen's General Matrix Progression sequence of infinite terms is given as:

$$T_n = k(a) + i(n-m) + f \tag{1.1}$$

Where $T_n$ is the $n^{th}$ term of the sequence, $k$ is the migration column value, $i$ is the migration row value, $a$ is the migration column factor, $m$ is the migration row factor, $n$ is the number of terms and $f$ is the first term. The formula for the value of the migration column factor, $a$ is obtained from [7]:

$$a = \frac{-1+\sqrt{8n-7}}{2} \quad \text{and } a = l-1 \tag{1.2}$$

Where $a$ is the migration column factor, $n$ is the number of terms and $l$ is the level of the term. Meanwhile, the formula for the value of the migration row factor, $m$ is attained using the equation presented by [8] which is given as:

$$m = \frac{a^2+a+2}{2} \tag{1.3}$$

Where $m$ is the migration row factor and $a$ is the migration row factor. The first term, $f$ is the value of the first member of Kifilideen's General Matrix Progression Sequence of infinite terms. Also, the first term, $f$ is the value of the first level and first step member in Kifilideen's Matrix Structural Framework [8]. The migration column value, $k$ is the difference between the value of the first member of one level and the value of the first member of the immediate previous level. The migration row value, $i$ is the difference between the value of a step in a level and the value of the immediate previous step in the same level [11].
The series of Kifilideen's General Matrix Progression of infinite terms is the summation of the values of collection in the progressive members of the sequence of a system or set. The member of the sequence of the system or set has endless terms where terms are progressive into levels and steps (within level) with increasing members set in successive levels and just one member in the first level. This kind of series is needed in determining the overall value(s) or worth of the collection of the progressive members of the system which is/are useful for budgeting, analyzing, accounting, allocating and planning the system of arrangement that adopts such series. This study invented and applied the epoch of Kifilideen's Sum Formula for Kifilideen's General Matrix Progression Series of infinite terms in solving real-world problems.

# 2 Materials and Methods

The mathematical induction of sum formulas of the series of bi - numbers product progression was formulated and established. The sum formulas generated were incorporated into inventing Kifilideen's Sum Formula for Kifilideen's General Matrix Progression Series of infinite terms.

## 2.1 Mathematical induction of sum formula for the series of bi–numbers product progression

$$S_n = 1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + 5 \times 6 + 6 \times 7 + \cdots + n[n+1] \quad (2.1)$$

$$S_n = \sum_{n=1}^{n} n[n+1] = \frac{n[n+1][n+2]}{3} \quad (2.2)$$

**Proof**

The mathematical induction of the sum formula (2.2) of the series of bi–numbers product progression is provided as follows:

$$S_1 = 1 \times 2 = [1+1] \quad (2.3)$$

$$S_2 = 1 \times 2 + 2 \times 3 = [1+1] + [2+2+2] \quad (2.4)$$

$$S_2 = \boxed{\begin{array}{ccccc} 1 & + & 1 & + & 1 \\ & & + & & \\ 2 & + & 2 & + & 2 \end{array}} - 1 \quad (2.5)$$

In (2.4), the addition of 1 appears two times [that is $1+1$] and the addition of 2 appears three times [that is $2+2+2$] but in (2.5), the rectangle contains the addition of three 1 and three 2. So, 1 has to be removed from the rectangle in (2.5) for (2.5) to be equal to (2.4). More so, the rectangle in (2.5) contains three $[1+2]$. Using the sum of arithmetic progression formula presented by [17] to solve each of the $[1+2]$, (2.5) can be generated as:

$$S_2 = 3 \times \frac{2 \times 3}{2} - 1 = 3 \times \frac{2 \times 3}{2} - {}^{3}_{0}C = 8$$

$$S_3 = 1 \times 2 + 2 \times 3 + 3 \times 4 = [1+1] + [2+2+2] + [3+3+3+3] \quad (2.7)$$

$$S_3 = \boxed{\begin{array}{ccccccc} 1 & + & 1 & + & 1 & + & 1 \\ & & & + & & & \\ 2 & + & 2 & + & 2 & + & 2 \\ & & & + & & & \\ 3 & + & 3 & + & 3 & + & 3 \end{array}} - \begin{array}{ccc} 1 & + & 1 \\ & + & \\ & 2 & \end{array} \quad (2.8)$$

In (2.7), the addition of 1 appears two times [that is $1+1$], the addition of 2 appears three times [that is $2+2+2$] and the addition of 3 appears four times [that is $3+3+3+3$] but in (2.8), the rectangle contains the addition of four 1, four 2 and four 3 altogether. So, $[1+1]$ and [2] have to be removed from the rectangle in (2.8) for (2.8) to be equal to (2.7). More so, the rectangle in (2.8) contains four $[1+2+3]$. Using the sum of arithmetic progression formula given by [3] to solve each of the $[1+2+3]$, (2.8) can be generated as:

$$S_3 = 4 \times \frac{3 \times 4}{2} - 4 = 4 \times \frac{3 \times 4}{2} - {}^{4}_{1}C = 20 \quad (2.9)$$

$$S_4 = 1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 = [1+1] + [2+2+2] + [3+3+3+3] + [4+4+4+4+4] \quad (2.10)$$

$$S_4 = \boxed{\begin{array}{c} 1+1+1+1+1 \\ 2+2+2+2+2 \\ 3+3+3+3+3 \\ 4+4+4+4+4 \end{array}} - \begin{array}{c} 1+1+1 \\ + \\ 2+2 \\ + \\ 3 \end{array} \tag{2.11}$$

In (2.10) the addition of 1 appears two times [that is $1+1$], the addition of 2 appears three times [that is $2+2+2$], the addition of 3 appears four times [that is $3+3+3+3$] and the addition of 4 appears five times [that is $4+4+4+4+4$] but in (2.11), the rectangle contains the addition of five 1, five 2, five 3 and five 4 altogether. So, [$1+1+1$], [$2+2$] and [3] have to be removed from the rectangle in (2.11) for (2.11) to be equal to (2.10). More so, the rectangle in (2.11) contains five [$1+2+3+4$]. Using the sum of arithmetic progression formula presented by [13] to solve each of the [$1+2+3+4$], (2.11) can be generated as:

$$S_4 = 5\times\frac{4\times5}{2} - 10 = 5\times\frac{4\times5}{2} - {}^{5}_{2}C = 40$$
(2.12)

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$$S_n = (n+1)\times\frac{n(n+1)}{2} - {}^{(n+1)}_{(n-2)}C \tag{2.13}$$
$$S_n = \frac{n(n+1)^2}{2} - {}^{(n+1)}_{(n-2)}C \tag{2.14}$$

In summary, the mathematical induction of the sum formula (2.2) of the series of bi-numbers product progression is provided as follows:

$$S_1 = 2\times\frac{1\times2}{2} - {}^{2}_{-1}C = 2 - \frac{2!}{-1!(2-(-1))!} = 2 - \frac{2!}{-1!3!} = 2 - \frac{2\times1\times0\times-1!}{-1!3!} = 2-0 \quad = 2 \tag{2.15}$$
$$S_2 = 3\times\frac{2\times3}{2} - 1 = 3\times\frac{2\times3}{2} - {}^{3}_{0}C = 8 \tag{2.16}$$
$$S_3 = 4\times\frac{3\times4}{2} - 4 = 4\times\frac{3\times4}{2} - {}^{4}_{1}C = 20 \tag{2.17}$$
$$S_4 = 5\times\frac{4\times5}{2} - 10 = 5\times\frac{4\times5}{2} - {}^{5}_{2}C = 40 \tag{2.18}$$

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$$S_n = (n+1)\times\frac{n(n+1)}{2} - {}^{(n+1)}_{(n-2)}C \tag{2.19}$$

$$S_n = \frac{n(n+1)^2}{2} - \frac{(n+1)!}{((n+1)-(n-2))!(n-2)!} \tag{2.20}$$

$$S_n = \frac{n(n+1)^2}{2} - \frac{(n+1)!}{(3)!(n-2)!} \tag{2.21}$$

$$S_n = \frac{n(n+1)^2}{2} - \frac{(n+1)(n)(n-1)(n-2)!}{(3)!(n-2)!} \tag{2.22}$$

$$S_n = \frac{n(n+1)^2}{2} - \frac{(n+1)(n)(n-1)}{(3)!} \tag{2.23}$$
$$S_n = \frac{n(n+1)^2}{2} - \frac{(n+1)(n)(n-1)}{6} \tag{2.24}$$

$$S_n = \frac{3n(n+1)^2-(n+1)(n)(n-1)}{6} \tag{2.25}$$

$$S_n = \frac{n(n+1)(3(n+1)-(n-1))}{6} \tag{2.26}$$
$$S_n = \frac{n(n+1)(3n+3-n+1\ )}{6} \tag{2.27}$$

$$S_n = \frac{n(n+1)(2n+4)}{6} \tag{2.28}$$
$$S_n = \frac{2n(n+1)(n+2)}{6} \tag{2.29}$$
$$S_n = \frac{n(n+1)(n+2)}{3} \tag{2.30}$$

Proved

Note that the transformation part of (2.16), (2.17), (2.18) and other series into combination are obtained from Pascal's triangle in Figure 2.1 as follows:

4th Diagonal

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & 2 & 1 & & \\
 & 1 & 3 & 3 & 1\,({}^{3}_{0}C) & & \\
1 & 4 & 6 & 4({}^{4}_{1}C) & 1 & & \\
1 & 5 & 10 & 10\,({}^{5}_{2}C) & 5 & 1 & \\
1 & 6 & 15 & 20({}^{6}_{3}C) & 15 & 6 & 1
\end{array}$$

Figure 2.1: Pascal's triangle

## 2.2 Mathematical Induction of Sum Formula for the Series of Bi–Similar Numbers Product Progression

$$S_n = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + \cdots + n^2 \tag{2.31}$$
$$S_n = \sum_{n=1}^{n} n^2 = \frac{n[n+1][2n+1]}{6} \tag{2.32}$$

Proof

The mathematical induction of the sum formula (2.32) of the series of bi–similar numbers product progression is provided as follows:

$$S_1 = 1^2 = 1 \tag{2.32}$$

$$S_2 = 1^2 + 2^2 = [1] + [2+2] \tag{2.33}$$

$$S_2 = \boxed{\begin{matrix} 1 + 1 \\ + \\ 2 + 2 \end{matrix}} - 1 \tag{2.34}$$

In (2.33) the addition of 1 appears once [that is 1] and the addition of 2 appears two times [that is 2 + 2] but in (2.34), the rectangle contains the addition of two 1 and two 2. So, 1 has to be removed from the rectangle in (2.34) for (2.34) to be equal to (2.33). More so, the rectangle in (2.34) contains two [1 + 2]. Using the sum of arithmetic progression formula presented by [14] Stroud and Booth 2007 to solve each of the [1 + 2], (2.34) can be generated as:

$$S_2 = 2 \times \frac{2\times 3}{2} - 1 = 2 \times \frac{2\times 3}{2} - {}^{3}_{0}C = 5 \tag{2.35}$$

$$S_3 = 1^2 + 2^2 + 3^2 = [1] + [2+2] + [3+3+3] \tag{2.36}$$

$$S_3 = \boxed{\begin{array}{c} 1 + 1 + 1 \\ + \\ 2 + 2 + 2 \\ + \\ 3 + 3 + 3 \end{array}} - \begin{array}{c} 1 + 1 \\ + \\ 2 \end{array} \quad (2.37)$$

In (2.36) the addition of 1 appears once [that is 1], the addition of 2 appears two times [that is 2 + 2] and the addition of 3 appears three times [that is 3 + 3 + 3] but in (2.37), the rectangle contains the addition of 1, three 2 and three 3 altogether. So, [1 + 1] and [2] have to be removed from the rectangle in (2.37) for (2.37) to be equal to (2.36). More so, the rectangle in (2.37) contains three [1 + 2 + 3]. Using the sum of arithmetic progression formula presented by [6] to solve each of the [1 + 2 + 3], (2.37) can be generated as:

$$S_3 = 3 \times \frac{3\times4}{2} - 4 = 3 \times \frac{3\times4}{2} - {}^{4}_{1}C = 14 \quad (2.38)$$

$$S_4 = 1^2 + 2^2 + 3^2 + 4^2 = [1] + [2+2] + [3+3+3] + [4+4+4+4] \quad (2.39)$$

$$S_4 = \boxed{\begin{array}{c} 1 + 1 + 1 + 1 \\ + \\ 2 + 2 + 2 + 2 \\ + \\ 3 + 3 + 3 + 3 \\ + \\ 4 + 4 + 4 + 4 \end{array}} - \begin{array}{c} 1 + 1 + 1 \\ + \\ 2 + 2 \\ + \\ 3 \end{array} \quad (2.40)$$

In (2.39) the addition of 1 appears once [that is 1], the addition of 2 appears two times [that is 2 + 2], the addition of 3 appears three times [that is 3 + 3 + 3] and the addition of 4 appears four times [that is 4 + 4 + 4 + 4] but in (2.40), the rectangle contains the addition of four 1, four 2, four 3 and four 4 altogether. So, [1 + 1 + 1], [2 + 2] and [3] have to be removed from the rectangle in (2.40) for (2.40) to be equal to (2.39). More so, the rectangle in (2.40) contains four [1 + 2 + 3 + 4]. Using the sum of arithmetic progression formula to solve each of the [1 + 2 + 3 + 4], (2.40) can be generated as:

$$S_4 = 4 \times \frac{4\times5}{2} - 10 = 4 \times \frac{4\times5}{2} - {}^{5}_{2}C = 30 \quad (2.41)$$

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$$S_n = (n) \times \frac{n(n+1)}{2} - {}^{(n+1)}_{(n-2)}C \quad (2.42)$$

$$S_n = \frac{n^2(n+1)}{2} - {}^{(n+1)}_{(n-2)}C \quad (2.43)$$

In summary, the mathematical induction of the sum formula (2.32) of the series of bi–similar numbers product progression is provided as follows:

$$S_1 = 1 \times \frac{1\times2}{2} - {}^{2}_{-1}C = 1 - \frac{2!}{-1!(2-(-1))!} = 1 - \frac{2!}{-1!3!} = 1 - \frac{2\times1\times0\times-1!}{-1!3!} = 1 - 0 = 1 \quad (2.44)$$

$$S_2 = 2 \times \frac{2\times3}{2} - 1 = 2 \times \frac{2\times3}{2} - {}^{3}_{0}C = 5 \quad (2.45)$$

$$S_3 = 3 \times \frac{3\times4}{2} - 4 = 3 \times \frac{3\times4}{2} - {}^{4}_{1}C = 14 \quad (2.46)$$

$$S_4 = 4 \times \frac{4\times5}{2} - 10 = 4 \times \frac{4\times5}{2} - {}^{5}_{2}C = 30 \quad (2.47)$$

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$$S_n = (n) \times \frac{n(n+1)}{2} - {}^{(n+1)}_{(n-2)}C \quad (2.48)$$

$$S_n = \frac{n^2(n+1)}{2} - \frac{(n+1)!}{((n+1)-(n-2))!(n-2)!} \quad (2.49)$$

$$S_n = \frac{n^2(n+1)}{2} - \frac{(n+1)!}{(3)!(n-2)!} \quad (2.50)$$

$$S_n = \frac{n^2(n+1)}{2} - \frac{(n+1)(n)(n-1)(n-2)!}{(3)!(n-2)!} \quad (2.51)$$

$$S_n = \frac{n^2(n+1)}{2} - \frac{(n+1)(n)(n-1)}{(3)!} \quad (2.52)$$

$$S_n = \frac{n^2(n+1)}{2} - \frac{(n+1)(n)(n-1)}{6} \quad (2.53)$$

$$S_n = \frac{3n^2(n+1)-(n+1)(n)(n-1)}{6} \quad (2.54)$$

$$S_n = \frac{n(n+1)(3(n)-(n-1))}{6} \quad (2.55)$$

$$S_n = \frac{n(n+1)(3n-n+1)}{6} \quad (2.56)$$

$$S_n = \frac{n(n+1)(2n+1)}{6} \quad (2.57)$$

Proved

**2.3 The Series of the Kifilideen's General Matrix Progression of Infinite Terms**

The series of Kifilideen's General Matrix Progression Sequence of infinite terms of increasing members set in successive columns and the first column having one member is given as:

$$S_n = [k(0) + i(0) + f] + [k(1) + i(0) + f] + [k(1) + i(1) + f] + [k(2) + i(0) + f] + [k(2) + i(1) + f] + [k(2) + i(2) + f] + [k(3) + i(0) + f] + [k(3) + i(1) + f] + [k(3) + i(2) + f] + [k(3) + i(3) + f] + \ldots, \ldots, \ldots, \quad (2.58)$$

The series of (2.59) is an infinite term. The arrangement of the terms of the series in Kifilideen's Matrix Structural Framework is given in Table 1.1. In Table 1.1, it can be shown that level 1 has 1 term, level 2 has two terms, level 3 has three terms and level 4 has four terms. Other higher levels in Kifilideen's Structural Framework for Kifilideen's General Matrix Progression Series of infinite terms follow the same trend. In level 1, the first member starts from step 1 and ends in step 1. In level 2, the first member starts from step 2 and ends in step 3; in level 3, the first member starts from step 3 and ends in step 5; and in level 4, the first member starts from step 4 and step 7. The higher levels follow the same progressive trend.

**2.4 Mathematical Induction of the Kifilideen's Sum Formula for the Kifilideen's General Matrix Progression Series of infinite terms**

The mathematical induction of the Kifilideen's Sum Formula for the Kifilideen's General Matrix Progression Series infinite terms is given as:

$$S_n = k(q_n) + i(w_n) + nf \quad (2.59)$$

Where

$$q_n = \frac{(a)(a+1)(a+2)+3a(n-m-a))}{3} \quad (2.60)$$

$$w_n = \frac{(a-1)(a)(a+1)+3(n-m)(n-m+1)}{6} \quad (2.61)$$

Where $S_n$ is the sum of the first $n^{th}$ terms of the series, $k$ is the migration column value, $i$ is the migration row value, $f$ is the first term, $n$ is the value of the number of terms to sum, $q_n$ is the sum migration column factor and $w_n$ is the sum migration row factor.

**Proof**

The Kifilideen's General Matrix Progression Series of infinite terms is given as:

$S_n = [k(0) + i(0) + f] + [k(1) + i(0) + f] + [k(1) + i(1) + f] + [k(2) + i(0) + f] + [k(2) + i(1) + f]$

$+ [k(2) + i(2) + f] + [k(3) + i(0) + f] + [k(3) + i(1) + f] + [k(3) + i(2) + f] + [k(3) + i(3) + f] +$
$\dots, \dots, \dots,$ (2.62)

Level 1, $l = 1$

$$S_1 = k((0)) + i((0)) + f = k(q_1) + i(w_1) + f \tag{2.63}$$

Level 2, $l = 2$

$$S_2 = k((0) + 1) + i((0) + 0) + 2f = k(q_2) + i(w_2) + 2f \tag{2.64}$$

$$S_3 = k((0) + (1 + 1)) + i\big((0) + (0 + 1)\big) + 3f = k(q_3) + i(w_3) + 3f \tag{2.65}$$

Level 3, $l = 3$

$$S_4 = k\big((0) + (1 + 1) + 2\big) + i\big((0) + (0 + 1) + 0\big) + 4f = k(q_4) + i(w_4) + 4f \tag{2.66}$$

$$S_5 = k\big((0) + (1 + 1) + 2 + 2\big) + i\big((0) + (0 + 1) + 0 + 1\big) + 5f = k(q_5) + i(w_5) + 5f \tag{2.67}$$

$$S_6 = k\big((0) + (1 + 1) + (2 + 2 + 2)\big) + i\big((0) + (0 + 1) + (0 + 1 + 2)\big) + 6f = k(q_6) + i(w_6) + 6f \tag{2.68}$$

Level 4, $l = 4$

$$S_7 = k\big((0) + (1 + 1) + (2 + 2 + 2) + 3\big) + i\big((0) + (0 + 1) + (0 + 1 + 2) + 0\big) + 7f$$

$$S_7 = k(q_7) + i(w_7) + 7f \tag{2.69}$$

$$S_8 = k\big((0) + (1 + 1) + (2 + 2 + 2) + 3 + 3\big) + i\big((0) + (0 + 1) + (0 + 1 + 2) + 0 + 1\big) + 8f$$

$$S_8 = k(q_8) + i(w_8) + 8f \tag{2.70}$$

$$S_9 = k\big((0) + (1 + 1) + (2 + 2 + 2) + 3 + 3 + 3\big) + i\big((0) + (0 + 1) + (0 + 1 + 2) + 0 + 1 + 2\big) + 9f$$

$$S_9 = k(q_9) + i(w_9) + 9f \tag{2.71}$$

$$S_{10} = k\big((0) + (1 + 1) + (2 + 2 + 2) + (3 + 3 + 3 + 3)\big)$$
$$+ i\big((0) + (0 + 1) + (0 + 1 + 2) + (0 + 1 + 2 + 3)\big) + 10f$$

$$S_{10} = k(q_{10}) + i(w_{10}) + 10f \tag{2.72}$$

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Level $l$, $l = l$

$$S_n = k(q_n) + i(w_n) + nf \tag{2.73}$$

Where $S_n$ is the sum of the first $n^{th}$terms of the series, $k$ is the migration column value, $i$ is the migration row value, $f$ is the first term, $n$ is the value of the number of terms to sum, $q_n$ is the sum migration column factor and $w_n$ is the sum migration row factor.

From the mathematical induction of (2.63) to (2.72), the mathematical induction of the sum migration column factor, $q_n$ is obtained as:

Level 1, $l = 1$

$$q_1 = (0) = 1 \times \frac{0 \times 1}{2} - {}_{-2}^{\;1}C - 0 \times 0 = 0 - \frac{1!}{(-2)!(1-(-2))!} - 0 = \frac{1!}{(-2)!3!} = \frac{1 \times 0 \times (-1) \times (-2)!}{(-2)!3!} = 0 \tag{2.73}$$

Level 2, $l = 2$

$$q_2 = (0) + 1 = 2 \times \frac{1 \times 2}{2} - {}_{-1}^{\;2}C - 1 \times 1 = 2 - \frac{2!}{(-1)!(2-(-1))!} - 1 = 1 - \frac{2!}{(-1)!3!} = 1 - \frac{2 \times 1 \times 0 \times (-1)!}{(-1)!3!} = 1 \tag{2.74}$$

$$q_3 = (0) + (1 + 1) = 2 \times \frac{1 \times 2}{2} - {}_{-1}^{\;2}C - 1 \times 0$$
$$= 2 - \frac{2!}{(-1)!(2-(-1))!} - 0 = 2 - \frac{2!}{(-1)!3!} = 2 - \frac{2 \times 1 \times 0 \times (-1)!}{(-1)!3!} = 2 \tag{2.75}$$

Level 3, $l = 3$

$$q_4 = (0) + (1 + 1) + 2 \tag{2.76}$$

$$q_4 = 0 + \begin{array}{c} 1 + 1 \\ + \\ 2 \end{array} = \boxed{\begin{array}{c} 1 + 1 + 1 \\ + \\ 2 + 2 + 2 \end{array}} - 1 - 2 \times 2 \tag{2.77}$$

In (2.76), the addition of 0 appears once [that is 0], the addition of 1 appears two times [that is 1 + 1] and the addition of 2 appears once [that is 2] but in (2.77), the rectangle contains the addition of three

1 and three 2 altogether. So, [1] and [2 × 2] have to be removed from the rectangle in (2.77) for (2.77) to be equal to (2.76). More so, the rectangle in (2.77) contains three [1 + 2]. Using the sum of the arithmetic progression formula presented by [2] to solve each of the [1 + 2], (2.77) can be generated as:

$$q_4 = 3 \times \frac{2\times3}{2} - 1 - 2 \times 2 = 3 \times \frac{2\times3}{2} - {}_0^3C - 2 \times 2 = 4 \tag{2.78}$$

$$q_5 = (0) + (1 + 1) + 2 + 2 \tag{2.79}$$

$$q_5 = 0 + \begin{array}{c} 1 + 1 \\ + \\ 2 + 2 \end{array} = \boxed{\begin{array}{c} 1 + 1 + 1 \\ + \\ 2 + 2 + 2 \end{array}} - 1 - 2 \times 1 \tag{2.80}$$

In (2.79), the addition of 0 appears once [that is 0], the addition of 1 appears two times [that is 1 + 1] and the addition of 2 appears two times [that is 2 + 2] but in (2.80), the rectangle contains the addition of three 1 and three 2 altogether. So, [1] and [2 × 1] have to be removed from the rectangle in (2.80) for (2.80) to be equal to (2.79). More so, the rectangle in (2.80) contains three [1 + 2]. Using the sum of arithmetic progression formula presented by [5] to solve each of the [1 + 2], (2.80) can be generated as:

$$q_5 = 3 \times \frac{2\times3}{2} - 1 - 2 \times 1 = 3 \times \frac{2\times3}{2} - {}_0^3C - 2 \times 1 = 6 \tag{2.81}$$

$$q_6 = (0) + (1 + 1) + (2 + 2 + 2) \tag{2.82}$$

$$q_6 = 0 + \begin{array}{c} 1 + 1 \\ + \\ 2 + 2 + 2 \end{array} = \boxed{\begin{array}{c} 1 + 1 + 1 \\ + \\ 2 + 2 + 2 \end{array}} - 1 - 2 \times 0 \tag{2.83}$$

In (2.82), the addition of 0 appears once [that is 0], the addition of 1 appears two times [that is 1 + 1] and the addition of 2 appears three times [that is 2 + 2 + 2] but in (2.83), the rectangle contains the addition of three 1 and three 2 altogether. So, [1] and [2 × 0] have to be removed from the rectangle in (2.83) for (2.83) to be equal to (2.82). More so, the rectangle in (2.83) contains three [1 + 2]. Using the sum of the arithmetic progression formula presented by [1] to solve each of the [1 + 2], (2.83) can be generated as:

$$q_6 = 3 \times \frac{2\times3}{2} - 1 - 2 \times 0 = 3 \times \frac{2\times3}{2} - {}_0^3C - 2 \times 0 = 8 \tag{2.84}$$

Level 4, $l = 4$

$$q_7 = (0) + (1 + 1) + (2 + 2 + 2) + 3 \tag{2.85}$$

$$q_7 = 0 + \begin{array}{c} 1 + 1 \\ + \\ 2 + 2 + 2 \\ + \\ 3 \end{array} = \boxed{\begin{array}{c} 1 + 1 + 1 + 1 \\ + \\ 2 + 2 + 2 + 2 \\ + \\ 3 + 3 + 3 + 3 \end{array}} - \begin{array}{c} 1 + 1 \\ + \\ 2 \end{array} - 3 \times 3 \tag{2.86}$$

In (2.85), the addition of 0 appears once [that is 0], the addition of 1 appears two times [that is 1 + 1], the addition of 2 appears three times [that is 2 + 2 + 2] and the addition of 3 appears once [that is 3] but in (2.86), the rectangle contains the addition of four 1, four 2 and four 3 altogether. So, [1 + 1], [2] and 3 × 3 have to be removed from the rectangle in (2.86) for (2.86) to be equal to (2.85). More so, the rectangle in (2.86) contains four [1 + 2 + 3]. Using the sum of arithmetic progression formula presented by [15] to solve each of the [1 + 2 + 3], (2.86) can be generated as:

$$q_7 = 4 \times \frac{3\times4}{2} - 4 - 3 \times 3 = 4 \times \frac{3\times4}{2} - {}_1^4C - 3 \times 3 = 11 \tag{2.87}$$

$$q_8 = (0) + (1+1) + (2+2+2) + 3 + 3 \tag{2.88}$$

$$q_8 = 0 + \begin{array}{c} 1+1 \\ + \\ 2+2+2 \\ + \\ 3+3 \end{array} = \boxed{\begin{array}{c} 1+1+1+1 \\ + \\ 2+2+2+2 \\ + \\ 3+3+3+3 \end{array}} - \begin{array}{c} 1+1 \\ + \\ 2 \end{array} - 3 \times 2 \tag{2.89}$$

In (2.88), the addition of 0 appears once [that is 0], the addition of 1 appears two times [that is 1 + 1], the addition of 2 appears three times [that is 2 + 2 + 2] and the addition of 3 appears two times [that is 3 + 3] but in (2.89), the rectangle contains the addition of four 1, four 2 and four 3 altogether. So, [1 + 1]. [2] and [3 × 2] has to be removed from the rectangle in (2.89) for (2.89) to be equal to (2.88). More so, the rectangle in (2.89) contains four [1 + 2 + 3]. Using the sum of arithmetic progression formula presented by [16] to solve each of the [1 + 2 + 3], (2.89) can be generated as:

$$q_8 = 4 \times \frac{3\times4}{2} - 4 - 3 \times 2 = 4 \times \frac{3\times4}{2} - {}^{4}_{1}C - 3 \times 2 = 14 \tag{2.90}$$

$$q_9 = (0) + (1+1) + (2+2+2) + 3 + 3 + 3 \tag{2.91}$$

$$q_9 = 0 + \begin{array}{c} 1+1 \\ + \\ 2+2+2 \\ + \\ 3+3+3 \end{array} = \boxed{\begin{array}{c} 1+1+1+1 \\ + \\ 2+2+2+2 \\ + \\ 3+3+3+3 \end{array}} - \begin{array}{c} 1+1 \\ + \\ 2 \end{array} - 3 \times 1 \tag{2.92}$$

In (2.91), the addition of 0 appears once [that is 0], the addition of 1 appears two times [that is 1 + 1], the addition of 2 appears three times [that is 2 + 2 + 2] and the addition of 3 appears three times [that is 3 + 3 + 3] but in (2.92), the rectangle contains the addition of four 1, four 2 and four 3 altogether. So, [1 + 1]. [2] and [3 × 1] has to be removed from the rectangle in (2.92) for (2.92) to be equal to (2.91). More so, the rectangle in (2.92) contains four [1 + 2 + 3]. Using the sum of arithmetic progression formula presented by [18] to solve each of the [1 + 2 + 3], (2.92) can be generated as:

$$q_9 = 4 \times \frac{3\times4}{2} - 4 - 3 \times 1 = 4 \times \frac{3\times4}{2} - {}^{4}_{1}C - 3 \times 1 = 17 \tag{2.93}$$

$$q_{10} = (0) + (1+1) + (2+2+2) + (3+3+3+3) \tag{2.94}$$

$$q_{10} = 0 + \begin{array}{c} 1+1 \\ + \\ 2+2+2 \\ + \\ 3+3+3+3 \end{array} = \boxed{\begin{array}{c} 1+1+1+1 \\ + \\ 2+2+2+2 \\ + \\ 3+3+3+3 \end{array}} - \begin{array}{c} 1+1 \\ + \\ 2 \end{array} - 3 \times 0 \tag{2.95}$$

In (2.94), the addition of 0 appears once [that is 0], the addition of 1 appears two times [that is 1 + 1], the addition of 2 appears three times [that is 2 + 2 + 2] and the addition of 3 appears four times [that is 3 + 3 + 3 + 3] but in (2.95), the rectangle contains the addition of four 1, four 2 and four 3 altogether. So, [1 + 1]. [2] and [3 × 0] has to be removed from the rectangle in (2.95) for (2.95) to be equal to (2.94). More so, the rectangle in (2.95) contains four [1 + 2 + 3]. Using the sum of arithmetic progression formula presented by [4] to solve each of the [1 + 2 + 3], (2.95) can be generated as:

$$S_{10} = 4 \times \frac{3\times4}{2} - 4 - 3 \times 0 = 4 \times \frac{3\times4}{2} - {}^{4}_{1}C - 3 \times 0 = 20 \tag{2.96}$$

In summary, the mathematical induction of the sum migration column factor formula from (2.73) to (2.96) is given as follows:

Level 1, $l = 1$

$$q_1 = 1 \times \frac{0\times1}{2} - {}_{-2}^{1}C - 0 \times 0 = 1 \times \frac{0\times1}{2} - {}_{-2}^{1}C - (1-1)\big((1-1)-0\big) = 1 \times \frac{0\times1}{2} - {}_{-2}^{1}C - (1-1)\big((1-1)-(1-1)\big) \tag{2.97}$$

Level 2, $l = 2$

$$q_2 = 2 \times \frac{1\times2}{2} - {}_{-1}^{2}C - 1 \times 1 = 2 \times \frac{1\times2}{2} - {}_{-1}^{2}C - (2-1)\big((2-1)-0\big) = 2 \times \frac{1\times2}{2} - {}_{-1}^{2}C - (2-1)\big((2-1)-(2-2)\big) \tag{2.98}$$

$$q_3 = 2 \times \frac{1\times2}{2} - {}_{-1}^{2}C - 1 \times 0 = 2 \times \frac{1\times2}{2} - {}_{-1}^{2}C - (2-1)\big((2-1)-1\big) = 2 \times \frac{1\times2}{2} - {}_{-1}^{2}C - (2-1)\big((2-1)-(3-2)\big) \tag{2.99}$$

Level 3, $l = 3$

$$q_4 = 3 \times \frac{2\times3}{2} - {}_{0}^{3}C - 2 \times 2 = 3 \times \frac{2\times3}{2} - {}_{0}^{3}C - (3-1)\big((3-1)-0\big) = 3 \times \frac{2\times3}{2} - {}_{0}^{3}C - (3-1)\big((3-1)-(4-4)\big) \tag{2.100}$$

$$q_5 = 3 \times \frac{2\times3}{2} - {}_{0}^{3}C - 2 \times 1 = 3 \times \frac{2\times3}{2} - {}_{0}^{3}C - (3-1)\big((3-1)-1\big) = 3 \times \frac{2\times3}{2} - {}_{0}^{3}C - (3-1)\big((3-1)-(5-4)\big) \tag{2.101}$$

$$q_6 = 3 \times \frac{2\times3}{2} - {}_{0}^{3}C - 2 \times 0 = 3 \times \frac{2\times3}{2} - {}_{0}^{3}C - (3-1)\big((3-1)-2\big) = 3 \times \frac{2\times3}{2} - {}_{0}^{3}C - (3-1)\big((3-1)-(6-4)\big) \tag{2.102}$$

Level 4, $l = 4$

$$q_7 = 4 \times \frac{3\times4}{2} - {}_{1}^{4}C - 3 \times 3 = 4 \times \frac{3\times4}{2} - {}_{1}^{4}C - (3-1)\big((3-1)-0\big) = 4 \times \frac{3\times4}{2} - {}_{1}^{4}C - (4-1)\big((4-1)-(7-7)\big) \tag{2.103}$$

$$q_8 = 4 \times \frac{3\times4}{2} - {}_{1}^{4}C - 3 \times 2 = 4 \times \frac{3\times4}{2} - {}_{1}^{4}C - (3-1)\big((3-1)-1\big) = 4 \times \frac{3\times4}{2} - {}_{1}^{4}C - (4-1)\big((4-1)-(8-7)\big) \tag{2.104}$$

$$q_9 = 4 \times \frac{3\times4}{2} - {}_{1}^{4}C - 3 \times 1 = 4 \times \frac{3\times4}{2} - {}_{1}^{4}C - (3-1)\big((3-1)-2\big) = 4 \times \frac{3\times4}{2} - {}_{1}^{4}C - (4-1)\big((4-1)-(9-7)\big) \tag{2.105}$$

$$q_{10} = 4 \times \frac{3\times4}{2} - {}_{1}^{4}C - 3 \times 0 = 4 \times \frac{3\times4}{2} - {}_{1}^{4}C - (3-1)\big((3-1)-2\big) = 4 \times \frac{3\times4}{2} - {}_{1}^{4}C - (4-1)\big((4-1)-(10-7)\big) \tag{2.106}$$

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Level $l$, $l = l$

$$q_n = l \times \frac{(l-1)\times l}{2} - {}_{(l-3)}^{l}C - (l-1)\big((l-1)-(n-m)\big) \tag{2.107}$$

$$q_n = \frac{l^2(l-1)}{2} - \frac{l!}{(l-(l-3))!(l-3)!} - (l-1)(l-1-n+m) \tag{2.108}$$

$$q_n = \frac{l^2(l-1)}{2} - \frac{l(l-1)(l-2)(l-3)!}{3!(l-3)!} + (l-1)(n-m-l+1) \tag{2.109}$$

$$q_n = \frac{l^2(l-1)}{2} - \frac{l(l-1)(l-2)}{3!} + (l-1)(n-m-l+1) \tag{2.110}$$

$$q_n = \frac{l^2(l-1)}{2} - \frac{l(l-1)(l-2)}{6} + (l-1)(n-m-l+1) \tag{2.111}$$

$$q_n = \frac{3l^2(l-1)-l(l-1)(l-2)}{6} + (l-1)(n-m-l+1) \tag{2.112}$$

$$q_n = \frac{l(l-1)\big(3l-(l-2)\big)}{6} + (l-1)(n-m-l+1) \tag{2.113}$$

$$q_n = \frac{l(l-1)(2l+2)}{6} + (l-1)(n-m-l+1) \tag{2.114}$$

$$q_n = \frac{l(l-1)(l+1)}{3} + (l-1)(n-m-l+1) \tag{2.115}$$

$$q_n = \frac{l(l-1)(l+1)}{3} + (l-1)(n-m-l+1) \tag{2.116}$$

$$q_n = \frac{l(l-1)(l+1)+3(l-1)(n-m-l+1)}{3} \tag{2.117}$$

$$q_n = \frac{(l-1)(l)(l+1)+3(l-1)(n-m-l+1))}{3} \tag{2.118}$$

From (1.2), $l = a + 1$ (2.119)

Therefore,

$$q_n = \frac{(a)(a+1)(a+2)+3a(n-m-a))}{3} \tag{2.120}$$

Where $q_n$ is the sum migration column factor, $a$ is the migration column factor, $n$ is the value of the number of terms to sum, $m$ is the migration row factor.

From the mathematical induction of (2.63) to (2.72), the mathematical induction of the sum migration row factor, $w_n$ is obtained as:

Level 1, $l = 1$

$$w_1 = (0) = {}_{-2}^{\;1}C = \frac{2!}{(-1)!(2-(-1))!} = \frac{2!}{(-1)!3!} = \frac{2\times1\times0\times(-1)!}{(-1)!3!} = 0 \tag{2.121}$$

Level 2, $l = 2$

$$w_2 = (0) + 0 = {}_{-1}^{\;2}C + 0 = \frac{2!}{(-1)!(2-(-1))!} + 0 = \frac{2!}{(-1)!3!} + 0 = \frac{2\times1\times0\times(-1)!}{(-1)!3!} + 0 = 0 \tag{2.122}$$

$$\begin{aligned} w_3 &= (0) + (0+1) = {}_{-1}^{\;2}C + (0+1) = \frac{2!}{(-1)!\left(2-(-1)\right)!} + (0+1) \\ &= \frac{2!}{(-1)!\,3!} + (0+1) = \frac{2\times1\times0\times(-1)!}{(-1)!\,3!} + (0+1) = 1 \end{aligned} \tag{2.123}$$

Level 3, $l = 3$

$$w_4 = (0) + (0+1) + 0 \tag{2.124}$$

$$w_4 = \begin{array}{l} 0 \\ + \\ 0 \;+\; 1 \\ + \\ 0 \end{array} = \begin{array}{c} 0 + 0 \\ + \\ 1 \end{array} + \quad 0 \tag{2.125}$$

The circle in (2.125) contains two 0 and one 1 altogether. Recall [1] is equivalent to, so, we have

$$w_4 = {}_{0}^{3}C + 0 = {}_{0}^{3}C + (4-4) \tag{2.126}$$

$$w_5 = (0) + (0+1) + 0 + 1 \tag{2.127}$$

$$w_5 = \begin{array}{l} 0 \\ + \\ 0 \;+\; 1 \\ + \\ 0 \;+\; 1 \end{array} = \begin{array}{c} 0 + 0 \\ + \\ 1 \end{array} + \quad 0 + 1 \tag{2.128}$$

The circle in (2.128) contains two 0 and one 1 altogether. Recall [1] is equivalent to ${}_{0}^{3}C$, so, we have:

$$w_5 = {}_{0}^{3}C + 0 + 1 = {}_{0}^{3}C + (4-4) + \ (5-4) \tag{2.128}$$

$$w_6 = (0) + (0+1) + (0+1+2) \tag{2.129}$$

$$w_6 = \begin{array}{l} 0 \\ + \\ 0 \;+\; 1 \\ + \\ 0 \;+\; 1 + 2 \end{array} = \begin{array}{c} 0 + 0 \\ + \\ 1 \end{array} + \quad 0 + 1 + 2 \tag{2.130}$$

The circle in (2.130) contain two 0 and one 1 altogether. Recall [1] is equivalent to ${}_{0}^{3}C$, so, we have:

$$w_6 = {}_{0}^{3}C + 0 + 1 + 2 = {}_{0}^{3}C + (4-4) + \ (5-4) + (6-4) \tag{2.131}$$

Level 4, $l = 4$

$$w_7 = (0) + (0+1) + (0+1+2) + 0 \tag{2.132}$$

$$w_7 = \begin{array}{l} 0 \\ + \\ 0 \;\;+\; 1 \\ + \\ 0 \;+\; 1 + 2 \\ + \\ 0 \end{array} = \begin{array}{c} 0 + 0 + 0 \\ + \\ 1 + 1 \\ + \\ 2 \end{array} + 0 \tag{2.133}$$

The circle in (2.133) contains three 0, two 1 and one 2 altogether. Recall [1 + 1] + [2] is equivalent to ${}_{1}^{4}C$, so we have:

$$w_7 = \ {}_{1}^{4}C + 0 = {}_{1}^{4}C + (7-7) \tag{2.134}$$

$$w_8 = (0) + (0+1) + (0+1+2) + 0 + 1 \tag{2.135}$$

$$w_8 = \begin{array}{l} 0 \\ + \\ 0 \; + 1 \\ + \\ 0 + 1 + 2 \\ + \\ 0 + 1 \end{array} = \begin{array}{c} 0 + 0 + 0 \\ + \\ 1 + 1 \\ + \\ 2 \end{array} + 0 + 1 \tag{2.136}$$

The circle in (2.137) contain three 0, two 1 and one 2 altogether. Recall $[1+1] + [2]$ is equivalent to ${}_1^4C$, so, we have:

$$w_8 = {}_1^4C + 0 + 1 = {}_1^4C + (7-7) + (8-7) \tag{2.137}$$

$$q_9 = (0) + (0+1) + (0+1+2) + 0 + 1 + 2 \tag{2.138}$$

$$w_9 = \begin{array}{l} 0 \\ + \\ 0 \; + 1 \\ + \\ 0 + 1 + 2 \\ + \\ 0 + 1 + 2 \end{array} = \begin{array}{c} 0 + 0 + 0 \\ + \\ 1 + 1 \\ + \\ 2 \end{array} + 0 + 1 + 2 \tag{2.139}$$

The circle in (2.139) contain three 0, two 1 and one 2 altogether. Recall $[1+1] + [2]$ is equivalent to ${}_1^4C$, so we have:

$$w_9 = {}_1^4C + 0 + 1 + 2 = {}_1^4C + (7-7) + (8-7) + (9-7) \tag{2.140}$$

$$w_{10} = (0) + (0+1) + (0+1+2) + (0+1+2+3) \tag{2.141}$$

$$w_{10} = \begin{array}{l} 0 \\ + \\ 0 \; + 1 \\ + \\ 0 + 1 + 2 \\ + \\ 0 + 1 + 2 + 3 \end{array} = \begin{array}{c} 0 + 0 + 0 \\ + \\ 1 + 1 \\ + \\ 2 \end{array} + 0 + 1 + 2 + 3 \tag{2.142}$$

The circle in (2.142) contain three 0, two 1 and one 2 altogether. Recall $[1+1] + [2]$ is equivalent to ${}_1^4C$, so we have:

$$w_{10} = {}_1^4C + 0 + 1 + 2 + 3 = {}_1^4C + (7-7) + (8-7) + (9-7) + (10-7) \tag{2.143}$$

$$\vdots$$

In summary, the mathematical induction of the sum migration column factor formula from (2.121) to (2.143) is given as follows:

Level 1, $\boldsymbol{l} = \mathbf{1}$

$$w_{\mathbf{1}} = {}_{-2}^{\mathbf{1}}C + 0 = {}_{-2}^{\mathbf{1}}C + \frac{0 \times \mathbf{1}}{2} = {}_{-2}^{\mathbf{1}}C + (1-1) = {}_{(\mathbf{1}-3)}^{\mathbf{1}}C + \frac{(\mathbf{1}-1)\times((\mathbf{1}-1)+1)}{2} \tag{2.144}$$

$$\mathbf{1} \times \frac{0\times\mathbf{1}}{2} - {}_{-2}^{\mathbf{1}}C - 0 \times 0 = \mathbf{1} \times \frac{0\times\mathbf{1}}{2} - {}_{-2}^{\mathbf{1}}C - (1-1)\big((\mathbf{1}-1) - 0\big) = \mathbf{1} \times \frac{0\times\mathbf{1}}{2} - {}_{-2}^{\mathbf{1}}C - (\mathbf{1}-1)\big((\mathbf{1}-1) - (\mathbf{1}-1)\big) \tag{2.145}$$

Level 2, $\boldsymbol{l} = \mathbf{2}$

$$w_{\mathbf{2}} = {}_{-1}^{\mathbf{2}}C + 0 = {}_{-1}^{\mathbf{2}}C + \frac{0\times 1}{2} = {}_{-1}^{\mathbf{2}}C + (2-2) = {}_{(\mathbf{2}-3)}^{\mathbf{2}}C + \frac{(\mathbf{2}-2)\times((\mathbf{2}-2)+1)}{2} \tag{2.146}$$

$$w_{\mathbf{3}} = {}_{-1}^{\mathbf{2}}C + 0 + 1 = {}_{-1}^{\mathbf{2}}C + \frac{1\times 2}{2} = {}_{-1}^{\mathbf{2}}C + (2-2) + (3-2) = {}_{(\mathbf{2}-3)}^{\mathbf{2}}C + \frac{(\mathbf{3}-2)\times((\mathbf{3}-2)+1)}{2} \tag{2.147}$$

Level 3, $l = 3$

$$w_4 = {}^{3}_{0}C + 0 = {}^{3}_{0}C + \frac{0\times1}{2} = {}^{3}_{0}C + (4-4) = {}^{3}_{(3-3)}C + \frac{(4-4)\times((4-4)+1)}{2} \quad (2.148)$$

$$w_5 = {}^{3}_{0}C + 0 + 1 = {}^{3}_{0}C + \frac{1\times2}{2} = {}^{3}_{0}C + (4-4) + (5-4) = {}^{3}_{(3-3)}C + \frac{(5-4)\times((5-4)+1)}{2} \quad (2.149)$$

$$w_6 = {}^{3}_{0}C + 0 + 1 + 2 = {}^{3}_{0}C + \frac{2\times3}{2}\, {}^{3}_{0}C + (4-4) + (5-4) + (6-4) = {}^{3}_{(3-3)}C + \frac{(6-4)\times((6-4)+1)}{2} \quad (2.150)$$

Level 4, $l = 4$

$$w_7 = {}^{4}_{1}C + 0 = {}^{4}_{1}C + \frac{0\times1}{2} = {}^{4}_{1}C + (7-7) = {}^{4}_{(4-3)}C + \frac{(7-7)\times((7-7)+1)}{2} \quad (2.151)$$

$$w_8 = {}^{4}_{1}C + 0 + 1 = {}^{4}_{1}C + \frac{1\times2}{2} = {}^{4}_{1}C + (7-7) + (8-7) = {}^{4}_{(4-3)}C + \frac{(8-7)\times((8-7)+1)}{2} \quad (2.152)$$

$$w_9 = {}^{4}_{1}C + 0 + 1 + 2 = {}^{4}_{1}C + \frac{2\times3}{2} = {}^{4}_{1}C + (7-7) + (8-7) + (9-7) = {}^{4}_{(4-3)}C + \frac{(9-7)\times((9-7)+1)}{2} \quad (2.153)$$

$$w_{10} = {}^{4}_{1}C + 0 + 1 + 2 + 3 = {}^{4}_{1}C + \frac{3\times4}{2} = {}^{4}_{1}C + (7-7) + (8-7) + (9-7) + (10-7) = {}^{4}_{(4-3)}C + \frac{(10-7)\times((10-7)+1)}{2} \quad (2.154)$$

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Level $l$, $l = l$

$$w_n = {}^{l}_{(l-3)}C + \frac{(n-m)(n-m+1)}{2} \quad (2.155)$$

$$w_n = \frac{l!}{(l-(l-3))!(l-3)!} + \frac{(n-m)(n-m+1)}{2} \quad (2.156)$$

$$w_n = \frac{l(l-1)(l-2)(l-3)!}{3!(l-3)!} + \frac{(n-m)(n-m+1)}{2} \quad (2.157)$$

$$w_n = \frac{l(l-1)(l-2)}{3!} + \frac{(n-m)(n-m+1)}{2} \quad (2.158)$$

$$w_n = \frac{l(l-1)(l-2)}{6} + \frac{(n-m)(n-m+1)}{2} \quad (2.159)$$

$$w_n = \frac{l(l-1)(l-2)+3(n-m)(n-m+1)}{6} \quad (2.160)$$

From (1.2), $l = a + 1$ (2.161)

Therefore,

$$w_n = \frac{(a+1)(a)(a-1)+3(n-m)(n-m+1)}{6} \quad (2.162)$$

$$w_n = \frac{(a-1)(a)(a+1)+3(n-m)(n-m+1)}{6} \quad (2.163)$$

Where $w_n$ is the sum migration row factor, $a$ is the migration column factor, $n$ is the value of the number of terms to sum, $m$ is the migration row factor. Table 2.1 shows the relationship between the migration column factor, $a$ and migration row factor, $m$ which is obtained from (2.144) to (2.154). From Table 2.1 we have:

$$m = {}^{a+1}_{a-1}C + 1 \quad (2.164)$$

$$m = \frac{(a+1)!}{((a+1)-(a-1))!(a-1)!} + 1 \quad (2.165)$$

$$m = \frac{(a+1)!}{(a+1-a+1)!(a-1)!} + 1 \quad (2.166)$$

$$m = \frac{(a+1)!}{2!(a-1)!} + 1 \quad (2.167)$$

$$m = \frac{(a+1)(a)(a-1)!}{2!(a-1)!} + 1 \quad (2.168)$$

$$m = \frac{(a+1)(a)}{2!} + 1 \quad (2.169)$$

$$m = \frac{a^2+a}{2} + 1 \quad (2.170)$$

$$m = \frac{a^2+a+2}{2} \quad (2.171)$$

So, the migration row factor is given as:

$$\text{migration row factor} = m = \frac{a^2+a+2}{2} \quad (2.172)$$

**Table 2.1:** The relationship between the migration column factor, $a$ and migration row factor, $m$ which

is obtained from (2.144) to (2.154).

| Levels, $l$ | Migration column factor, $a$ | Migration row factor, $m$ |
|---|---|---|
| 1 | **0** | $1 = 0 + 1 = {}_{-1}^{1}C + 1 = {}_{0-1}^{0+1}C + 1$ |
| 2 | **1** | $2 = 1 + 1 = {}_{0}^{2}C + 1 = {}_{1-1}^{1+1}C + 1$ |
| 3 | **2** | $4 = 3 + 1 = {}_{1}^{3}C + 1 = {}_{2-1}^{2+1}C + 1$ |
| 4 | **3** | $7 = 6 + 1 = {}_{2}^{4}C + 1 = {}_{3-1}^{3+1}C + 1$ |
| 5 | **4** | $11 = 10 + 1 = {}_{3}^{5}C + 1 = {}_{4-1}^{4+1}C + 1$ |
| 6 | **5** | $16 = 15 + 1 = {}_{4}^{6}C + 1 = {}_{5-1}^{5+1}C + 1$ |
| . | . | . |
| . | . | . |
| . | . | . |
| $l$ | $\boldsymbol{a} = l - 1$ | $m = {}_{\boldsymbol{a}-1}^{\boldsymbol{a}+1}C + 1$ |

In summary, the mathematical induction of the Kifilideen's Sum Formula of the Kifilideen's General Matrix Progression Series of infinite terms is given as:

$$S_n = k(q_n) + i(w_n) + nf \quad (2.173)$$

Where

$$q_n = \frac{(a)(a+1)(a+2) + 3a(n-m-a))}{3} \quad (2.174)$$

$$w_n = \frac{(a-1)(a)(a+1) + 3(n-m)(n-m+1)}{6} \quad (2.175)$$

$$m = \frac{a^2+a+2}{2} \quad (2.176)$$

Where $S_n$ is the sum of the first $n^{th}$terns of the series, $k$ is the migration column value, $i$ is the migration row value, $f$ is the first term, $n$ is the value of the number of terms to sum, $q_n$ is the sum migration column factor, $w_n$ is the sum migration row factor, migration column factor, $a$ and migration row factor, $m$.

## 3 Results

### 3.1 Application of Epoch of Kifilideen's Sum Formula for Kifilideen's General Matrix Progression Series

The application of the epoch of Kifilideen's Sum Formula for Kifilideen's General Matrix Progression Series of infinite terms in solving real-world problems is presented as follows:

[a] A battery manufacturing company, say in Nigeria, may produce a series of batteries using Kifilideen's Matrix Progression Sequence concept. The levels of the batteries produced serve as the various weights of the batteries manufactured in order. Each level (weight grade) has sub-level (s) or step(s). The manufacturing company designs the charge capacity, $Ah$ of the battery as the sublevel(s) or steps property for each level (weight grade) of the battery. If the sequence of the batteries manufactured using the Kifilideen's matrix progression sequence of infinite terms is given as charge capacity $38Ah$ for level 1 of weight 200 $g$; charge capacities $43Ah$, $46Ah$ for level 2 of weight 400 $g$ each; charge capacities $48Ah$, $51Ah$, $54Ah$ for level 3 of weight 600 $g$ each; ...
Determine the following
(i) the sum of the charge capacities of batteries manufactured for the first twelve terms of the series of batteries produced by the company (ii) the sum of the charge capacities of batteries produced for the first hundred terms of the series of batteries manufactured by the company (iii) draw the Kifilideen's Matrix Structural Framework for the first five levels of the sequence of batteries manufacture by the company.

**Solution**

(i) For the first twelve terms of the series of batteries produced by the company, $n = 12$ (3.1)

Migration level factor of the charge capacity of the battery $= a = \frac{-1+\sqrt{8n-7}}{2}$ (3.2)

Migration level factor of the charge capacity of the battery $= a = \frac{-1+\sqrt{8\times 12-7}}{2}$ (3.3)
Migration level factor of the charge capacity of the battery $= a = \mathbf{4}.2170$ (3.4)
Migration level factor of the charge capacity of the battery $= a = 4$ (3.5)
Migration step factor of the charge capacity of the battery $= m = \frac{a^2+a+2}{2}$ (3.6)
Migration step factor of the charge capacity of the battery $= m = \frac{4^2+4+2}{2}$ (3.7)
Migration step factor of the charge capacity of the battery $= m = 11$ (3.8)
$f$ = charge capacity of the first term of the battery produced by the company $T_1 = 38Ah$ (3.9)
$k$ = the difference between the value of charge capacity of the battery of the first member of one level and the value of charge capacity of the battery of the first member of the immediate previous level = $T_2 - T_1$ (3.10)
$k = 43Ah - 38Ah = 5Ah$ (3.11)
$i$ = the difference between the value of charge capacity of the battery of a step in a level and the value charge capacity of the battery of the immediately previous step in the same level $= T_3 - T_2 = 46Ah - 43Ah = 3Ah$ ( 3.12)
Sum migration level factor of the charge capacity of the battery
$q_n = \frac{(a)(a+1)(a+2)+3a(n-m-a))}{3}$ (3.13)
Sum migration level factor of the charge capacity of the battery
$q_{12} = \frac{(4)(4+1)(4+2)+3\times 4(12-11-4))}{3}$ (3.14)
Sum migration level factor of the charge capacity of the battery
$q_{12} = \frac{(4)(5)(6)+3\times 4(-3))}{3}$ (3.15)
Sum migration level factor of the charge capacity of the battery
$q_{12} = 28$ (3.16)
Sum migration step factor of the charge capacity of the battery
$w_n = \frac{(a-1)(a)(a+1)+3(n-m)(n-m+1)}{6}$ (3.17)
Sum migration step factor of the charge capacity of the battery $w_{12} = \frac{(4-1)(4)(4+1)+3(12-11)(12-11+1)}{6}$
(3.18)
Sum migration step factor of the charge capacity of the battery
$w_{12} = \frac{(3)(4)(5)+3(1)(2)}{6}$ (3.19)
Sum migration step factor of the charge capacity of the battery
$w_{12} = 11$ (3.20)
$S_n = k(q_n) + i(w_n) + nf$ (3.21)
The sum of the charge capacities of batteries manufactured for the first twelve terms of the series of batteries produced by the company
$S_{12} = 5 \times (28) + 3(11) + 12 \times 38$ (3.22)
The sum of the charge capacities of batteries manufactured for the first twelve terms of the series of batteries produced by the company
$S_{12} = 629Ah$ (3.23)
(ii) For the first hundred terms of the series of batteries produced by the company,
$n = 100$ (3.24)
Migration level factor of the charge capacity of the battery
$a = \frac{-1+\sqrt{8n-7}}{2}$ (3.25)
Migration level factor of the charge capacity of the battery
$a = \frac{-1+\sqrt{8\times 100-7}}{2}$ (3.26)
Migration level factor of the charge capacity of the battery
$a = \mathbf{13}.5801$ (3.27)
Migration level factor of the charge capacity of the battery
$a = 13$ (3.28)
Migration step factor of the charge capacity of the battery

$m = \frac{a^2+a+2}{2}$ (3.29)

Migration step factor of the charge capacity of the battery

$m = \frac{13^2+13+2}{2}$ (3.30)

Migration step factor of the charge capacity of the battery

$m = 92$ (3.31)

$f$ = charge capacity of the first term of the battery produced by the company

$T_1 = 38Ah$ (3.32)

$k$ = the difference between the value of charge capacity of the battery of the first member of one level and the value of charge capacity of the battery of the first member of the immediate previous level = $T_2 - T_1$ (3.33)

$k = 43Ah - 38Ah = 5Ah$ (3.34)

$i$ = the difference between the value of charge capacity of the battery of a step in a level and the value charge capacity of the battery of the immediately previous step in the same level = $T_3 - T_2 = 46Ah - 43Ah = 3Ah$ ( 3.35)

Sum migration level factor of the charge capacity of the battery

$q_n = \frac{(a)(a+1)(a+2)+3a(n-m-a))}{3}$ (3.36)

Sum migration level factor of the charge capacity of the battery

$q_{100} = \frac{(13)(13+1)(13+2)+3\times13(100-92-13))}{3}$ (3.37)

Sum migration level factor of the charge capacity of the battery

$q_{100} = \frac{(13)(14)(15)+3\times4(-5))}{3}$ (3.38)

Sum migration level factor of the charge capacity of the battery

$q_{100} = 845$ (3.39)

Sum migration step factor of the charge capacity of the battery

$w_n = \frac{(a-1)(a)(a+1)+3(n-m)(n-m+1)}{6}$ (3.40)

Sum migration step factor of the charge capacity of the battery

$w_{100} = \frac{(13-1)(13)(13+1)+3(100-92)(100-92+1)}{6}$ (3.41)

Sum migration step factor of the charge capacity of the battery

$w_{100} = \frac{(12)(13)(14)+3(8)(9)}{6}$ (3.42)

Sum migration step factor of the charge capacity of the battery

$w_{100} = 400$ (3.43)

$S_n = k(q_n) + i(w_n) + nf$ (3.44)

The sum of the charge capacities of batteries produced for the first hundred terms of the series of batteries manufactured by the company

$S_{100} = 5 \times (845) + 3(400) + 100 \times 38$ (3.45)

The sum of the charge capacities of batteries produced for the first hundred terms of the series of batteries manufactured by the company

$S_{100} = 9225Ah$ (3.46)

(iii) Since migration level value = $k = 5BG$, migration step value = $i = 3Ah$ and the charge capacity of the first term of the battery produces is $38Ah$, then the Kifilideen's Matrix Structural Framework for the sequence of the first five levels of the batteries produce is given in Table 3.1.

**Table 3.1:** Kifilideen's Matrix Structural Framework for the sequence of questions (a)

| | $Levels(Weight, g)$ | | | | |
|---|---|---|---|---|---|
| Weights/Charge Capacity | Level 1, $l_1$ $W_1 = 200\ g$ | Level 1, $l_1$ $W_2 = 400\ g$ | Level 1, $l_1$ $W_3 = 600\ g$ | Level 1, $l_1$ $W_4 = 800\ g$ | Level 1, $l_1$ $W_5 = 1000\ g$ |
| Steps (Charge $s_1$ | $T_1 = 38Ah$ | | | | |
| $s_2$ | | $T_2 = 43Ah$ | | | |
| $s_3$ | | $T_3 = 46Ah$ | $T_4 = 48Ah$ | | |
| $s_4$ | | | $T_5 = 51Ah$ | $T_7 = 53Ah$ | |
| $s_5$ | | | $T_6 = 54Ah$ | $T_8 = 56Ah$ | $T_{11} = 58Ah$ |
| $s_6$ | | | | $T_9 = 59Ah$ | $T_{12} = 61Ah$ |
| $s_7$ | | | | $T_{10} = 62Ah$ | $T_{13} = 64Ah$ |
| $s_8$ | | | | | $T_{14} = 67Ah$ |
| $s_9$ | | | | | $T_{15} = 70Ah$ |

From Table 3.1, the value of the charge capacity of the battery in level 1 of weight 200 $g$ in step 1 is $38Ah$. The value of the charge capacities of batteries in level 2 of weight 400 $g$ in steps 2 and 3 are $43Ah$ and $46Ah$ respectively. More so, the value of the charge capacities of batteries in level 3 of weight 600 $g$ in steps 3, 4 and 5 are $48Ah$, $51Ah$ and $54Ah$ respectively. Furthermore, the value of the charge capacities of batteries in level 4 of weight 800 $g$ in steps 4, 5, 6 and 7 are $53Ah$, $56Ah$, $59Ah$ and $62Ah$ respectively. Meanwhile, the value of the charge capacities of batteries in level 5 of weight 1000 $g$ in steps 5, 6, 7, 8 and 9 are $58Ah$, $61Ah$, $64Ah$, $67Ah$ and $70Ah$ respectively. From Table 3.1 the sum of the charge capacities of batteries manufactured for the first twelve terms of the series of batteries produced by the company, $S_{12}$, is also $629Ah$. The difference in the value of the charge capacities of batteries in the last step in one level and the last step in the previous level is equivalent to the sum of the migration level value, $k$ and the migration step value, $i$. For this design, the difference in the value of the charge capacities of batteries in the last step in one level and the last step in the previous level is $8Ah$ (this is the difference in the value of the charge capacities of batteries in the last step in one level and the last step in the previous level $k + i = 5 + 3 = 8Ah$).

[b] A phone manufacturing company utilizes Kifilideen's Matrix Progression sequence of the infinite term to produce a series of phones. The levels 1, 2, 3, … of the phone are designed into various models 1, 2, 3, … in order. As the model advances, the physical appearance of the phone also advances. Each level or model of the phone has sub-level(s) or step(s). The storage capacity of the phone is the design property for each sub-level(s) or step(s) of each level or model of the phone. The storage capacity of the phone is measured in Gigabytes, GB. Given that the value of the storage capacities of phones manufactured by the company at the 5th and 10th terms of Kifilideen's General Matrix Progression Sequence of infinite terms are $15GB$ and $23GB$ and if the sum of the storage capacities of the first thirteen terms of the series of the manufactured phone collectively is $207GB$. Determine the following: (i) migration level value of the phone, $k$ (ii) migration step value of the phone, $i$ (iii) the storage capacity of the first term of the phone produce, $f$ (iv) the storage capacity of the 8th term of the phone produce (v) the sum of the storage capacity of the first six terms of the phone produce (vi) generate the Kifilideen's Matrix Structural Framework for the sequence of the first five levels or models.

**Solution**

(i) For the 5th term of the series of phones produced by the manufacturing company,

$$n = 5,\ T_5 = 15BG \tag{3.47}$$

Migration level factor of the storage capacity of the phone

$$a = \frac{-1+\sqrt{8n-7}}{2} \tag{3.48}$$

Migration level factor of the storage capacity of the phone

$$a = \frac{-1+\sqrt{8\times 5-7}}{2} \tag{3.49}$$

Migration level factor of the storage capacity of the phone
$a = \mathbf{2}.3723$ (3.50)
Migration level factor of the storage capacity of the phone
$a = 2$ (3.51)
Migration step factor of the storage capacity of the phone
$m = \frac{a^2+a+2}{2}$ (3.52)
Migration step factor of the storage capacity of the phone
$m = \frac{2^2+2+2}{2}$ (3.53)
Migration step factor of the storage capacity of the phone
$m = 4$ (3.54)
$T_n = k(a) + i(n - m) + f$ (3.55)
The storage capacity of the fifth term of the phone
$T_5 = k(2) + i(5 - 4) + f = 15BG$ (3.56)
$2k + i + f = 15BG$ (3.57)
For the 10th term of the series of phones produce by the manufacturing company,
$n = 10, T_{10} = 23BG$ (3.58)
Migration level factor of the storage capacity of the phone
$a = \frac{-1+\sqrt{8n-7}}{2}$ (3.59)
Migration level factor of the storage capacity of the phone
$a = \frac{-1+\sqrt{8\times10-7}}{2}$ (3.60)
Migration level factor of the storage capacity of the phone
$a = \mathbf{3}.7720$ (3.61)
Migration level factor of the storage capacity of the phone
$a = 3$ (3.62)
Migration step factor of the storage capacity of the phone
$m = \frac{a^2+a+2}{2}$ (3.63)
Migration step factor of the storage capacity of the phone
$m = \frac{3^2+3+2}{2}$ (3.64)
Migration step factor of the storage capacity of the phone
$m = 7$ (3.65)
$T_n = k(a) + i(n - m) + f$ (3.66)
The storage capacity of the tenth term of the phone
$T_{10} = k(3) + i(10 - 7) + f = 23BG$ (3.67)
$3k + 3i + f = 23BG$ (3.68)
For the first thirteen terms of the series of phone produce by the company,
$n = 13, \ S_{13} = 207BG$ (3.69)
Migration level factor of the storage capacity of the phone
$a = \frac{-1+\sqrt{8n-7}}{2}$ (3.70)
Migration level factor of the storage capacity of the phone
$a = \frac{-1+\sqrt{8\times13-7}}{2}$ (3.71)
Migration level factor of the storage capacity of the phone
$a = \mathbf{4}.4244$ (3.72)
Migration level factor of the storage capacity of the phone
$a = 4$ (3.73)
Migration step factor of the storage capacity of the phone
$m = \frac{a^2+a+2}{2}$ (3.74)
Migration step factor of the storage capacity of the phone
$m = \frac{4^2+4+2}{2}$ (3.75)

Migration step factor of the storage capacity of the phone

$m = \ \ 11$ (3.76)

Sum migration level factor of the storage capacity of the phone

$q_n = \frac{(a)(a+1)(a+2)+3a(n-m-a))}{3}$ (3.77)

Sum migration level factor of the storage capacity of the phone

$q_{13} = \frac{(4)(4+1)(4+2)+3\times4(13-11-4))}{3}$ (3.78)

Sum migration level factor of the storage capacity of the phone

$q_{13} = \frac{(4)(5)(6)+3\times4(-2))}{3}$ (3.79)

Sum migration level factor of the storage capacity of the phone

$q_{13} = 32$ (3.80)

Sum migration step factor of the storage capacity of the phone

$w_n = \frac{(a-1)(a)(a+1)+3(n-m)(n-m+1)}{6}$ (3.81)

Sum migration step factor of the storage capacity of the phone

$w_{13} = \frac{(4-1)(4)(4+1)+3(13-11)(13-11+1)}{6}$ (3.82)

Sum migration step factor of the storage capacity of the phone

$w_{13} = \frac{(3)(4)(5)+3(2)(3)}{6}$ (3.83)

Sum migration step factor of the storage capacity of the phone

$w_{13} = 13$ (3.84)

$S_n = k(q_n) + i(w_n) + nf$ (3.85)

The sum of the storage capacities of the first thirteen terms of the series of the manufactured phone collectively produces

$S_{13} = k \times (32) + i(13) + 13 \times f = 207BG$ (3.86)

$32k + 13i + 13f = 207BG$ (3.87)

From (3.57), (3.68) and (3.87), we have:

$2k + i + f = 15BG$ (3.88)

$3k + 3i + f = 23BG$ (3.89)

$32k + 13i + 13f = 207BG$ (3.90)

Multiply (3.88) with 13

$26k + 13i + 13f = 195BG$ (3.91)

Subtract (3.88) from (3.89), we have:

$k + 2i = 8BG$ (3.92)

Subtract (3.91) from (3.90)

$6k = 12BG$ (3.93)

$k = 2BG$ (3.94)

Put (3.94) in (3.92)

$k + 2i = 8BG$ (3.95)

$2 + 2i = 8BG$ (3.96)

$i = 3BG$ (3.97)

Put (3.95) and (3.97) in (3.88)

$2k + i + f = 15BG$ (3.98)

$2(2) + 3 + f = 15BG$ (3.99)

$f = 8BG$ (3.100)

So, $k = 2BG, i = 3BG \ and \ f = 8BG$ (3.101)

Migration level value of the phone, $k = 2BG$ (3.102)

(ii) Migration step value of the phone, $i = 3BG$ (3.103)

(ii) The storage capacity of the first term of the phone produces,

$f = 8BG$ (3.104)

(iii) For the 8th term of the series of phones produced by the manufacturing company,

$n \ = \ 8,$ (3.105)

Migration level factor of the storage capacity of the phone

$$a = \frac{-1+\sqrt{8n-7}}{2} \tag{3.106}$$

Migration level factor of the storage capacity of the phone

$$a = \frac{-1+\sqrt{8\times 8-7}}{2} \tag{3.107}$$

Migration level factor of the storage capacity of the phone

$$a = \mathbf{3}.2749 \tag{3.108}$$

Migration level factor of the storage capacity of the phone

$$a = 3 \tag{3.109}$$

Migration step factor of the storage capacity of the phone

$$m = \frac{a^2+a+2}{2} \tag{3.110}$$

Migration step factor of the storage capacity of the phone

$$m = \frac{3^2+3+2}{2} \tag{3.111}$$

Migration step factor of the storage capacity of the phone

$$m = 7 \tag{3.112}$$

$$T_n = k(a) + i(n-m) + f \tag{3.113}$$

Since $k = 2BG, i = 3BG\ and\ f = 8BG$ (3.114)

The storage capacity of the 8th term of the series of phones produces

$$T_8 = 2(3) + 3(8-7) + 8 = 17BG \tag{3.115}$$

The storage capacity of the 8th term of the series of phones produces

$$T_8 = 17BG \tag{3.116}$$

(iv) For the first six terms of the series of phone produce by the company,

$n = 6,$ (3.117)

Migration level factor of the storage capacity of the phone

$$a = \frac{-1+\sqrt{8n-7}}{2} \tag{3.118}$$

Migration level factor of the storage capacity of the phone

$$a = \frac{-1+\sqrt{8\times 6-7}}{2} \tag{3.119}$$

Migration level factor of the storage capacity of the phone

$$a = \mathbf{2}.7016 \tag{3.120}$$

Migration level factor of the storage capacity of the phone

$$a = 2 \tag{3.121}$$

Migration step factor of the storage capacity of the phone

$$m = \frac{a^2+a+2}{2} \tag{3.122}$$

Migration step factor of the storage capacity of the phone

$$m = \frac{2^2+2+2}{2} \tag{3.123}$$

Migration step factor of the storage capacity of the phone

$$m = 4 \tag{3.124}$$

Sum migration level factor of the storage capacity of the phone

$$q_n = \frac{(a)(a+1)(a+2)+3a(n-m-a))}{3} \tag{3.125}$$

Sum migration level factor of the storage capacity of the phone

$$q_6 = \frac{(2)(2+1)(2+2)+3\times 4(6-4-2))}{3} \tag{3.126}$$

Sum migration level factor of the storage capacity of the phone

$$q_6 = \frac{(2)(3)(4)+3\times 2(0))}{3} = \frac{24}{3} \tag{3.127}$$

Sum migration level factor of the storage capacity of the phone

$$q_6 = 8 \tag{3.128}$$

Sum migration step factor of the storage capacity of the phone

$$w_n = \frac{(a-1)(a)(a+1)+3(n-m)(n-m+1)}{6} \tag{3.129}$$

Sum migration step factor of the storage capacity of the phone

$$w_6 = \frac{(2-1)(2)(2+1)+3(6-4)(6-4+1)}{6} \tag{3.130}$$

Sum migration step factor of the storage capacity of the phone

$$w_6 = \frac{(1)(2)(3)+3(2)(3)}{6} \quad (3.131)$$

Sum migration step factor of the storage capacity of the phone

$$w_6 = 4 \quad (3.132)$$

$$S_n = k(q_n) + i(w_n) + nf \quad (3.133)$$

Since $k = 2BG, i = 3BG \; and \; f = 8BG$ (3.134)

The sum of the storage capacities of the first six terms of the phone produces

$$S_6 = 2 \times (8) + 3(4) + 6 \times 8 \quad (3.135)$$

Sum of the storage capacities of the first six terms of the phone produces

$$S_6 = 76BG \quad (3.136)$$

(vi) Since migration level value = $k = 2BG$, migration step value = $i = 3BG$ and the storage capacity of the first term of the phone produces, $f = 8BG$ then the Kifilideen's Matrix Structural Framework for the sequence of the first five levels of the phones manufacture is given in Table 3.2.

Table 3.2: Kifilideen matrix structural framework for the sequence of question (b)

| Level (Physical appearance), $l$ | | | | |
|---|---|---|---|---|
| $l_1$ | $l_2$ | $l_3$ | $l_4$ | $l_5$ |
| $T_1 = 8BG$ | | | | |
| | $T_2 = 10BG$ | | | |
| | $T_3 = 13BG$ | $T_4 = 12BG$ | | |
| | | $T_5 = 15BG$ | $T_7 = 14BG$ | |
| | | $T_6 = 18BG$ | $T_8 = 17BG$ | $T_{11} = 16BG$ |
| | | | $T_9 = 20BG$ | $T_{12} = 19BG$ |
| | | | $T_{10} = 23BG$ | $T_{13} = 22BG$ |
| | | | | $T_{14} = 25BG$ |
| | | | | $T_{15} = 28BG$ |

From Table 3.2, the value of the storage capacity of the phone in level 1 of step 1 is $8BG$. The value of the storage capacities of phones in level 2 of steps 2 and 3 are $10BG$ and $13BG$ respectively. More so, the value of the storage capacities of phones in level 3 of steps 3, 4 and 5 are $12BG$, $15BG$ and $18BG$ respectively. Furthermore, the value of the storage capacities of phones in level 4 of steps 4, 5, 6 and 7 are $14BG$, $17BG$, $20BG$ and $23BG$ respectively. Meanwhile, the value of the storage capacities of phones in level 5 of steps 5, 6, 7, 8 and 9 are 16 $BG$, $19BG$, $22BG$, $25BG$ and $28BG$ respectively. From table 3.2 the sum of the storage capacities of phone manufacturers for the first six terms of the series of phones produced by the company, $S_6$, is also $76BG$ and the sum of the storage capacities of phone manufacture for the first six terms of the series of phones produced by the company, $S_{13}$, is also $207BG$. The difference in the value of the storage capacities of phones in the last step in one level and the last step in the previous level is equivalent to the sum of the migration level value, $k$ and the migration step value, $i$. For this design, the difference in the value of the storage capacities of phones in the last step in one level and the last step in the previous level is $5BG$ (this is the difference in the value of the storage capacities of phones in the last step in one level and the last step in the previous level $k + i = 2 + 3 = 5BG$).

(c) A drink industry designs different grades of drinks based on taste level. Each grade of drinks concocted has step(s). Each grade's step(s) is/are determined by the litres of drink. The design of the litres of the drink concocted is formulated using Kifilideen's Matrix Progression Sequence of infinite terms. If the sum of litres of drink for the $n$ terms of the series of Kifilideen's General Matrix Progression Sequence of infinite terms generated by the company is 223 $litres$ and the last term of the series is found in level 4 of the Kifilideen's Matrix Structural Framework. If the migration level value of the drink, $k$, migration step value of the drink, $i$ and the litres of the first term of the drink, $f$ are 10, 5 and 2 $litres$ respectively. (i) Determine the number of terms in the series (ii) generate Kifilideen's Matrix Structural

Framework for the sequence of the first four levels of the drink.

**Solution**

For the first n terms of the series of drinks concocted by the industry,

$n = n, \ \ S_n = 223\ litres$ (3.137)

$Level = l = 4$ (3.138)

Migration level factor of the litres of drink $= a = l - 1 = 4 - 1 = 3$ (3.139)

Migration step factor of the litres of drink $= m = \frac{a^2+a+2}{2}$ (3.140)

Migration step factor of the litres of drink $= m = \frac{3^2+3+2}{2}$ (3.141)

Migration step factor of the litres of drink $= m = 7$ (3.142)

Sum migration level factor of the litre of the drink. $= q_n = \frac{(a)(a+1)(a+2)+3a(n-m-a))}{3}$ (3.143)

Sum migration level factor of the litre of the drink $= q_n = \frac{(3)(3+1)(3+2)+3\times3(n-7-3))}{3}$ (3.144)

Sum migration level factor of the litre of the drink $= q_n = \frac{(3)(4)(5)+3\times3(n-10))}{3} = \frac{24}{3}$ (3.145)

Sum migration level factor of the litre of the drink $= q_n = \frac{60+9(n-10))}{3}$ (3.146)

Sum migration level factor of the litre of the drink $= q_n = \frac{60+9n-90}{3}$ (3.147)

Sum migration level factor of the litre of the drink$= q_n = \frac{9n-30}{3}$ (3.148)

Sum migration level factor of the litre of the drink $= q_n = 3n - 10$ (3.149)

Sum migration step factor of the litre of the drink $= w_n = \frac{(a-1)(a)(a+1)+3(n-m)(n-m+1)}{6}$ (3.150)

Sum migration step factor of the litre of the drink$= w_n = \frac{(3-1)(3)(3+1)+3(n-7)(n-7+1)}{6}$ (3.151)

Sum migration step factor of the litre of the drink$= w_n = \frac{(2)(3)(4)+3(n-7)(n-6)}{6}$ (3.152)

Sum migration step factor of the litre of the drink $= w_n = \frac{24+3(n^2-13n+42)}{6}$ (3.153)

Sum migration step factor of the litre of the drink $= w_n = \frac{8+(n^2-13n+42)}{2}$ (3.154)

Sum migration step factor of the litre of the drink

$w_n = \frac{n^2-13n+50}{2}$ (3.155)

$S_n = k(q_n) + i(w_n) + nf$ (3.156)

Since $k = 10litres, i = 5litres\ and\ f = 2litres$ (3.157)

Sum of the litres of the first n terms of the drink concocted

$S_n = 10 \times (3n - 10) + 5\left(\frac{n^2-13n+50}{2}\right) + n \times 2 = 223litres$ (3.158)

$20 \times (3n - 10) + 5(n^2 - 13n + 50) + 4n = 446$ (3.159)

$5n^2 - n + 50 - 446 = 0$ (3.160)

$5n^2 - n - 396 = 0$ (3.161)

Using a quadratic formula, we have:

$n = \frac{-(-1)\pm\sqrt{(-1)^2-4\times5\times-396}}{2\times5} = \frac{1\pm\sqrt{7921}}{10} = \frac{1\pm89}{10} = \frac{90}{10}\ or\ \frac{-8.8}{10}$ (3.162)

$n = 9\ or - 8.8$ (3.163)

Since $n$ is positive, then:

$number\ of\ terms\ of\ the\ series\ of\ litres\ of\ drinks\ concocted = n = 9$ (3.164)

(ii) Since migration level value $= k = 10litres$, migration step value $= i = 5litres$ and the litres of the first term of the drink concocted $= f = 2litres$, then the Kifilideen's Matrix Structural Framework for the sequence of the first four levels of drinks concocted is given in Table 3.3.

**Table 3.3:** Kifilideen's Matrix Structural Framework for the sequence of question (c)

| Level, $l$ (Taste level) | | | |
|---|---|---|---|
| $l_1$ | $l_2$ | $l_3$ | $l_4$ |
| $T_1 = 2litres$ | | | |
| | $T_2 = 12litres$ | | |
| | $T_3 = 17litres$ | $T_4 = 22litres$ | |
| | | $T_5 = 27litres$ | $T_7 = 32litres$ |
| | | $T_6 = 32litres$ | $T_8 = 37litres$ |
| | | | $T_9 = 42litres$ |
| | | | $T_{10} = 47litres$ |

From Table 3.3, the value of the litres of the drink in level 1 of step 1 is $2litres$. The value of the litres of drink in level 2 of steps 2 and 3 are $12litres$ and $17litres$ respectively. More so, the value of the litres of drinks in level 3 of steps 3, 4 and 5 are $22litres$, $27litres$ and $32litres$ respectively. Furthermore, the value of the litres of drink in level 4 of steps 4, 5, 6 and 7 are $32litres$, $37litres$, $42litres$ and $47litres$ respectively. From the Table 3.3 the sum of the litres of drink concocted for the first nine terms of the series of drinks concocted by the industry, $S_9$, is also $223litres$. The difference in the value of the litres of drink in the last step in one level and the last step in the previous level is equivalent to the sum of the migration level value, $k$ and the migration step value, $i$. For this design, the difference in the value of the litres of drink in the last step in one level and the last step in the previous level is $15litres$ (this is the difference in the value of the litres of drink in the last step in one level and the last step in the previous level $k + i = 10 + 5 = 15litres$).

(d) A gold manufacturing company implements Kifilideen's Matrix Progression Sequence to produce gold into various masses. If the sum of masses of gold of the first four hundred terms of Kifilideen's Matrix Progression Sequence of infinite terms is $16424g$ and, the migration steps value of the gold, $i$ and the mass of the first term of the gold, $f$ are $-2g$ and $5g$ respectively. (i) Find the migration level value of the gold, $k$ of the series (ii) present the Kifilideen's Matrix Structural Framework for the sequence of the first three levels of the gold.

**Solution**

For the first four hundred terms of the series of gold produced by the company,

$n = 400, \ S_{400} = 16424$ (3.165)

Migration level factor of the masses of gold = $a = \frac{-1+\sqrt{8n-7}}{2}$ (3.166)

Migration level factor of the masses of gold = $a = \frac{-1+\sqrt{8\times400-7}}{2}$ (3.167)

Migration level factor of the masses of gold = $a = \mathbf{27}.7533$ (3.168)

Migration level factor of the masses of gold = $a = 27$ (3.169)

Migration step factor of the masses of gold = $m = \frac{a^2+a+2}{2}$ (3.170)

Migration step factor of the masses of gold = $m = \frac{27^2+27+2}{2}$ (3.171)

Migration step factor of the masses of gold = $m = 379$ (3.172)

Sum migration level factor of the masses of the gold. = $q_n = \frac{(a)(a+1)(a+2)+3a(n-m-a))}{3}$ (3.173)

Sum migration level factor of the masses of the gold

$q_{400} = \frac{(27)(27+1)(27+2)+3\times27(400-379-27))}{3}$ (3.174)

Sum migration level factor of the masses of the gold = $q_{400} = \frac{(27)(28)(29)+3\times27(-6))}{3} = \frac{24}{3}$ (3.175)

Sum migration level factor of the masses of the gold = $q_{400} = 7146$ (3.176)

Sum migration step factor of the masses of the gold = $w_n = \frac{(a-1)(a)(a+1)+3(n-m)(n-m+1)}{6}$ (3.177)

Sum migration step factor of the masses of the gold

$w_{400} = \frac{(27-1)(27)(27+1)+3(400-379)(400-379+1)}{6}$ (3.178)

Sum migration step factor of the masses of the gold $= w_{400} = \frac{(26)(27)(28)+3(21)(22)}{6}$ (3.179)

Sum migration step factor of the masses of the gold $= w_{400} = 3507$ (3.180)

$S_n = k(q_n) + i(w_n) + nf$ (3.181)

$S_{400} = k(q_{400}) + i(w_{400}) + 400 \times f$ (3.182)

Since $i = -2g$ $and$ $f = 5g$ (3.183)

The sum of the masses of the first four hundred terms of the gold manufacture

$S_{400} = k \times (7146) + (-2)(3507) + 400 \times 5 = 16424g$ (3.184)

$k \times (7146) = 16424 - 400 \times 5 + 2 \times 3507$ (3.185)

$7146k = 21438$ (3.186)

$k = \frac{21438}{7146}$ (3.187)

$k = 3g$ (3.188)

Migration level value of the masses of gold = k $= 3g$ (3.189)

(ii) Since migration level value $= k = 3g$ , migration step value $= i = -2g$ and the masses of the first term of the gold manufacture $= f = 5g$, then the Kifilideen's Matrix Structural Framework for the sequence of the first three levels of gold manufacture is given in Table 3.4.

Table 3.4: Kifilideen's Matrix Structural Framework for the sequence of question (d)

| | $level, l$ | |
|---|---|---|
| $l_1$ | $l_2$ | $l_3$ |
| $5g$ | | |
| | $8g$ | |
| | $6g$ | $11g$ |
| | | $9g$ |
| | | $7g$ |

From Table 3.4, the value of the masses of the gold in level 1 of step 1 is $5g$. The value of the masses of gold in level 2 of steps 2 and 3 are $8g$ and $6g$ respectively. More so, the value of the masses of gold in level 3 of steps 3, 4 and 5 are $11g$, $9g$ and $7g$ respectively. The difference in the value of the masses of gold in the last step in one level and the last step in the previous level is equivalent to the sum of the migration level value, $k$ and the migration step value, $i$. For this design, the difference in the value of the masses of gold in the last step in one level and the last step in the previous level is $1g$ (this is the difference in the value of the masses of gold in the last step in one level and the last step in the previous level $k + i = 3 + (-2) = 1g$).

**4 Conclusion**

This study invented and applied the epoch of Kifilideen's Sum Formula for Kifilideen's general matrix progression series of infinite terms to solve real-world problems. The mathematical induction of sum formulas of the bi–numbers product progression series was formulated and established. These sum formulas obtained were incorporated into inventing Kifilideen's sum formula for Kifilideen's general matrix progression series of infinite terms. The Kifilideen's sum formula invented in this paper and Kifilideen's components formulas of the Kifilideen's general matrix progression sequence of infinite terms were used in conjunction to proffer solutions to real-world problems. The established Kifilideen's sum formula for Kifilideen's general matrix progression series of infinite terms provides an easy and fastening process of finding the summation and evaluation of the overall value(s) of collection of progressive members of the Kifilideen's general matrix progression series of infinite terms.

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### Conflict of interest

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## References

[1] J. Bird, Engineering Mathematics, Linacre House, Jordan Hill, Oxford OX2 8DP 200 Wheeler Road, Burlington MA 01803, Ontario, Canada, North American: Newnes, Oxford, An imprint of Elsevier Science, 2003. https://www.amazon.com/Engineering-Mathematics-sixth-Bird-John/dp/B00BTM1OYO

[2] P. Brown, Sequences and Series: A Guide for Teachers (Years 11 -12), Education Service Australia Publisher, Carlton South Vic 3053, Austalia, 2013. https://www.amsi.org.au/ESA_Senior_Years/PDF/Sequences1d.pdf

[3] B.D. Bunday and H. Mulholland, Pure Mathematics for Advanced Level, McGraw Hill, United State: Elsevier Science Publisher, 2014. https://books.google.com.ng/books/about/Pure_Mathematics_for_Advanced_Level.html?id=02_iBQAAQBAJ

[4] J. Fowler and B. Snapp, Sequences and Series, Mooculus Massive Open Online Calculus, Department of Mathematics, University of Aegean, Mediterranean Sea, 2014. http://www.ia801507.us.archive.org/30/items/mooculus-sequencea-and-series.pdf

[5] K. Jones, *The Topic of Sequences and Series in the Curriculum and Textbooks for Schools in England: A Way to Link Number, Algebra and Geometry [Paper Presentation]*, International Conference on School Mathematics Textbooks, East China Normal University, Shanghai, China, (2011). https://www.eprints.soton.ac.uk/445735/1/JonesK_sequences_series_England_ICSMT_2011,pdf

[6] M.F. Macrae, A.O. Kalejaiye, Z.I. Chima, G.U. Garba, and M.O. Ademosu, New General Mathematics for Senior Secondary Schools 2, Harlow, England: Longman Publisher, 2001. http://www.amazon.com/New_General_Mathematics_for_Senior_Secondary_Schools

[7] K.L. Osanyinpeju, *Development of Kifilideen Trinomial Theorem Using Matrix Approach*, International Journal of Innovations in Engineering Research and Technology, **7**(7) (2020a), 117-135. https://doi.org/10.5281/zenodo.5794742

[8] K.L. Osanyinpeju, Utilization of Kifilideen Trinomial based on Matrix Approach in Computation of the Power of Base of Tri-Digits Number, International Conference on Electrical Engineering Applications (ICEEA 2020), Department of Electrical Engineering, Ahmadu Bello University, Zaria, Kaduna State, September 23-25 Nigeria, (2020b). https://doi.org/10.5281/zenodo.5804915

[9] K.L. OSANYINPEJU, *Inauguration of Negative Power of* $-n$ *of Kifilideen Trinomial Theorem using Standardized and Matrix Methods,* Acta Electronica Malaysia, **5**(1) (2021), 17-23. https://uilspace.unilorin.edu.ng/handle/20.500.12484/7407.

[10] K.L. OSANYINPEJU, *Inauguration of Kifilideen Theorem of Matrix Transformation of Negative Power of* $-n$ *of Trinomial Expression in which Three Variables x, y and z are Found in Parts of the Trinomial Expression with some other Applications*, Selforganizology, **9**(1-2) (2022a), 17-34. https://uilspace.unilorin.edu.ng/handle/20.500.12484/7406

[11] K.L. OSANYINEPJU, *Derivation of Formulas of the Components of Kifilideen Trinomial Expansion of Positive Power of* $\boldsymbol{N}$ *with other Developments*, Journal of Science, Technology, Mathematics and Education (JOSTMED), **18**(1) (2022b), 77-97. https://www.doi.org/10.5281/zenodo.12596949

[12] K.L. OSANYINPEJU, *Generation of Kifilideen's Generalized Matrix Progression Sequence of Infinite Term*, Ethiopian Journal of Science and Sustainable Development, **10**(2) (2023), 74-99. https://www.doi.org/10.20372/ejssdastu:v10.i2.2023.710

[15] E.A.S. PONTES, R.C.G. SILVA, E.G. PONTES and L.M. SILVA, *A Generality of Mathematics Thought by Carl Friedrich Gauss through his Contributions to Science*, American Journal of Engineering Research, 9(4) (2020), 246-252. http://www.ajer.org/papers/Vol-9-issue-4/ZC0904246252

[13] K. RICE and P. SCOTT, Carl Friedrich Gauss. Amt, **61**(4) (2005), 1-5. https://files.eric.ed.gov/fulltext/EJ743572

[14] K.A. STROUD and D.J. BOOTH, Engineering Mathematics, London, UK: Palgrave Publisher, 2007. https://www.amazon.com/Engineering-Mathematics-K-Stroud/dp/1403942463

[16] J.F. TALBERT, A. GODMAN and G.E.O. OGUM, Additional Mathematics for West Africa, United State: Prentice Hall Publisher, 1995. https://www.worldcat.org/title/additional-mathematics-for-west-africa-with-answers/oclc/12693055

[17] M.R. TUTTUH – ADEGUN, and M.G. ADEGOKE, New Further Mathematics Project 1. Bounty Press Limited, Odutola Orita Challenge Police Station, Mapo, Ibadan, 2014. http://www.subom.ng/2021/09/further-mathematics-project-1-by-tuttuh.html?m=1

[18] A.D. WITTMANN, Carl Friedrich Gauss and the Gauss Society: A brief overview. History of Geo – and Space Sciences, **11**(2020), 199-205. http://doi.org/10.5194/hgss-11-199-2020