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Exponential stabilization of an Euler-Bernoulli beam under boundary control

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Abstract. We study the free vibration of an Euler-Bernoulli beam without internal damping. By applying suitable control at the free boundary, we can exponentially dampen these vibrations. The exponential stability was proven using the Lyapunov method, and the results were confirmed through numerical simulation.

Keywords: Exponential stabilization, Euler-Bernoulli beam, boundary control, Lyapunov method.

2020 Mathematics Subject Classification: 93D23, 93D20, 35G46.

1 Introduction

We consider the following Euler-Bernoulli beam with boundary control:

$$\begin{cases} \rho w_{tt} + EIw_{xxxx} - Tw_{xx} = 0, & x \in (0, L), t \geq 0, \\ w(0,t) = w_x(0,t) = w_{xx}(L,t) = 0, & \\ -EIw_{xxx}(L,t) + Tw_x(L,t) = U(t), & t \geq 0, \\ w(x,0) = w^0(x), \quad w_t(x,0) = w^1(x), & x \in (0, L), \end{cases} \tag{1.1}$$

where $w(x,t)$ represents the displacements transverse of the beam at the position x for time t, ρ, L, T and EI are respectively uniform mass, length, axial tension and bending stiffness of the beam, and $U(t)$ is the boundary control.

Recently, the problem of beam control has become one of the most important research topics because of their applications in various fields. As it is one of the most effective mechanisms to achieve the stability of dynamic systems in the absence of internal dampers. We mention some of the works that adopted boundary control [2–5, 11]. For problems with internal damping, see [1, 6–10, 12].

The main contribution of this work is to achieve an exponential stabilization result by adopting boundary control only.

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2 Preliminary

In this section, we will define energy as well as suggest an ideal control to achieve exponential stability.

We introduce the energy associated with (1.1) as

$$E(t) = \frac{\rho}{2}\int_0^L w_t^2 dx + \frac{EI}{2}\int_0^L w_{xx}^2 dx + \frac{T}{2}\int_0^L w_x^2 dx. \tag{2.1}$$

Observe that this is the usual classical energy. The first term represents kinetic energy and the other terms represent potential energy.

Control

The control's objective is to reduce the free transverse vibrations of the beam. Lyapunov's direct method is used to construct a suitable boundary control at the free end of the beam.

To stabilize system (1.1), we propose the following control:

$$U(t) = -kw_t(L,t), \tag{2.2}$$

where k is a positive constant.

Lemma 2.1. *The energy functional* (2.1) *satisfies*

$$E'(t) = U(t)w_t(L,t), \qquad \forall t \geq 0. \tag{2.3}$$

Proof. We take the derivation of energy $E(t)$ with respect to time t, we have

$$E'(t) = \rho\int_0^L w_{tt}w_t dx + EI\int_0^L w_{xxt}w_{xx}dx + T\int_0^L w_{xt}w_x dx. \tag{2.4}$$

By integrating by parts the last two terms into the previous relation and taking into account the boundary condition (1.1), we get

$$E'(t) = \rho\int_0^L w_{tt}w_t dx + EI\left(\int_0^L w_t w_{xxxx}dx - w_t w_{xxx}(L,t)\right) - T\left(\int_0^L w_t w_{xx}dx - w_t w_x(L,t)\right). \tag{2.5}$$

Using the equations (1.1) into (2.5), we obtain

$$\begin{aligned} E'(t) &= \int_0^L \left(\rho w_{tt} + EIw_{xxxx} - Tw_{xx}\right) w_t dx + \left(Tw_x(L,t) - EIw_t(L,t)\right) w_t(L,t) \\ &= U(t)w_t(L,t). \end{aligned} \tag{2.6}$$

□

Remark 2.2. The proposed control (2.2) and Lemma 2.1. ensure that energy is dissipated. That is,

$$E'(t) = -kw_t^2(L,t) \leq 0, \qquad \forall t \geq 0. \tag{2.7}$$

3 Exponential stability

In order to prove the energy decay result, let us define the Lyapunov functional by

$$\mathcal{L}(t) = \epsilon E(t) + V(t), \tag{3.1}$$

where ϵ is a positive constant and

$$V(t) = \rho \int_0^L x w_x w_t dx.$$

Proposition 3.1. *There exist two positive constants τ_1 and τ_2, such that*

$$\tau_1 E(t) \leq \mathcal{L}(t) \leq \tau_2 E(t), \qquad \forall t \geq 0. \tag{3.2}$$

Proof. Applying Young's inequality, we get

$$\begin{aligned} |V(t)| &\leq \frac{L\rho}{2}\int_0^L w_t^2 dx + \frac{L\rho}{2}\int_0^L w_x^2 dx \\ &\leq \max\{L, \frac{L}{T}\} E(t). \end{aligned} \tag{3.3}$$

Considering $\epsilon > \max\{L, \frac{L}{T}\}$, we get (3.2) with $\tau_1 = \epsilon - \max\{L, \frac{L}{T}\}$ and $\tau_2 = \epsilon + \max\{L, \frac{L}{T}\}$. □

Lemma 3.2. *The derivative of $V(t)$ satisfies*

$$\begin{aligned} V'(t) &\leq -\frac{\rho}{2}\int_0^L w_t^2 dx - \frac{3EI}{2}\int_0^L w_{xx}^2 dx - \frac{T}{2}\int_0^L w_x^2 dx \\ &+ \frac{\rho\delta + 2}{2\delta} L w_t^2(L,t) - \frac{2T - \delta}{2} L w_x^2(L,t), \end{aligned} \tag{3.4}$$

where δ is a positive constant.

Proof. By differentiating $V(t)$ and using the first equation of (1.1), we have

$$\begin{aligned} V'(t) &= \rho \int_0^L (x w_{xt} w_t + x w_x w_{tt})\, dx \\ &= \int_0^L (\rho x w_{xt} w_t - EI x w_x w_{xxxx} + T x w_x w_{xx})\, dx. \end{aligned} \tag{3.5}$$

By integrating by parts the terms of (3.5) with respect for the boundary condition (1.1), as follows

$$\rho \int_0^L x w_{xt} w_t dx = \rho \frac{L}{2} w^2(L,t) - \frac{\rho}{2}\int_0^L w_t^2 dx, \tag{3.6}$$

$$\begin{aligned} EI \int_0^L x w_x w_{xxxx} dx &= EIL w_x w_{xxx}(L,t) - EI \int_0^L (w_x + x w_{xx})\, w_{xxx} dx \\ &= EIL w_x w_{xxx}(L,t) + \frac{3EI}{2}\int_0^L w_{xx}^2 dx \end{aligned} \tag{3.7}$$

and

$$T\int_0^L x w_x w_{xx} dx = \frac{TL}{2} w_x^2(L,t) - \frac{T}{2}\int_0^L w_x^2 dx. \tag{3.8}$$

By collecting results (3.5)-(3.8), we get

$$\begin{aligned} V(t) &= -\frac{\rho}{2}\int_0^L w_t^2 dx - \frac{3EI}{2}\int_0^L w_{xx}^2 dx - \frac{T}{2}\int_0^L w_x^2 dx \\ &\quad + \frac{L\rho}{2} w_t^2(L,t) - EILw_x w_{xxx}(L,t) + \frac{TL}{2} w_x^2(L,t) \\ &= -\frac{\rho}{2}\int_0^L w_t^2 dx - \frac{3EI}{2}\int_0^L w_{xx}^2 dx - \frac{T}{2}\int_0^L w_x^2 dx \\ &\quad + \frac{L\rho}{2} w_t^2(L,t) + LU(t)w_x(L,t) - \frac{TL}{2} w_x^2(L,t). \end{aligned} \tag{3.9}$$

Using boundary control (2.2) and Young's inequality, we obtain (3.4). □

Theorem 3.3. *The energy $E(t)$ satisfies along the solution of system (1.1)*

$$E(t) \le Ce^{-\lambda t}, \qquad t \ge 0, \tag{3.10}$$

where C and λ are two positive constants.

Proof. The derivative of $\mathcal{L}(t)$ is

$$\mathcal{L}'(t) = E'(t) + V'(t), \tag{3.11}$$

Taking Lemma (2.2) and Lemma (2.3), we obtain

$$\begin{aligned} \mathcal{L}'(t) &\le -\frac{\rho}{2}\int_0^L w_t^2 dx - \frac{3EI}{2}\int_0^L w_{xx}^2 dx - \frac{T}{2}\int_0^L w_x^2 dx \\ &\quad - \frac{2\delta\epsilon k - \rho\delta - 2}{2\delta} L w_t^2(L,t) - \frac{2TL - L\delta}{2} w_x^2(L,t). \end{aligned} \tag{3.12}$$

By choosing $\delta < 2T$ and $\epsilon > \frac{\rho\delta+2}{2\delta k} > \frac{\rho T+2}{2Tk}$ such that

$$2\delta\epsilon k - \rho\delta - 2 > 0,$$

and

$$2TL - L\delta > 0.$$

The result (3.12) becomes

$$\mathcal{L}'(t) \le -E(t), \qquad t \ge 0.$$

Using equivalence relation (3.2), we obtain

$$\mathcal{L}'(t) \le -\frac{1}{\tau_2}\mathcal{L}(t), \tag{3.13}$$

integrate this differential inequality over $(0,t)$, we have

$$\mathcal{L}(t) \le \mathcal{L}(0) e^{-\frac{1}{\tau_2}t}. \tag{3.14}$$

Using relation (3.2) again, we obtain (3.10) with $C = \frac{\mathcal{L}(0)}{\tau_1}$ and $\lambda = \frac{1}{\tau_2}$. □

4 Numerical simulations

In order to verify the effectiveness of the proposed control law, a simulation of the problem (1.1) described in this section was performed using the finite difference method. It should be noted that large values of the model parameters require very small time steps, which increases the computation time. Therefore, it is possible to extend the simulation parameters to improve the temporal performance. System parameters are listed in the following table:

Parameter	Description	Value
l	Length of the beam	$2m$
T	Tension	$10N$
ρ	The mass per unit length	$10kg/m$
EI	Bending stiffness	$10Nm^2$

The corresponding initial conditions are given as $w(x,0) = \cos(4\pi x)$ and $w_t(x,0) = 0$.

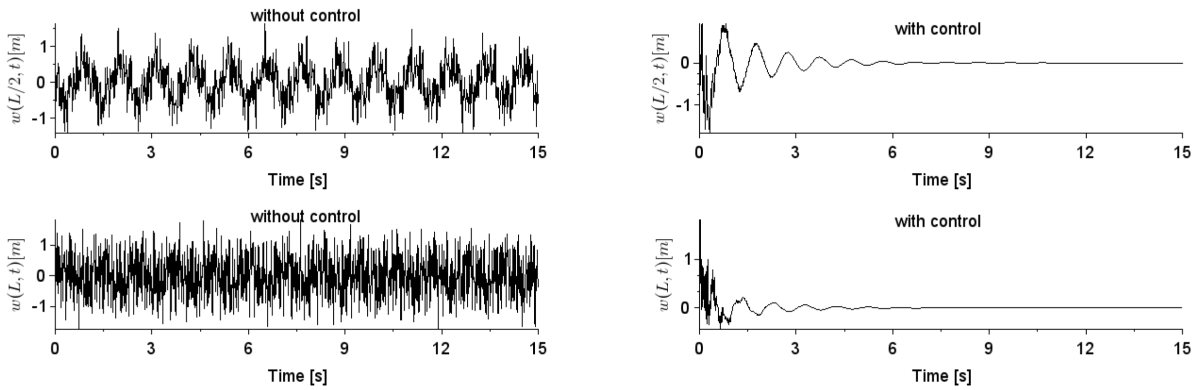

Figure 4.1: The displacement $w(x,t)$ of beam: without control and with control.

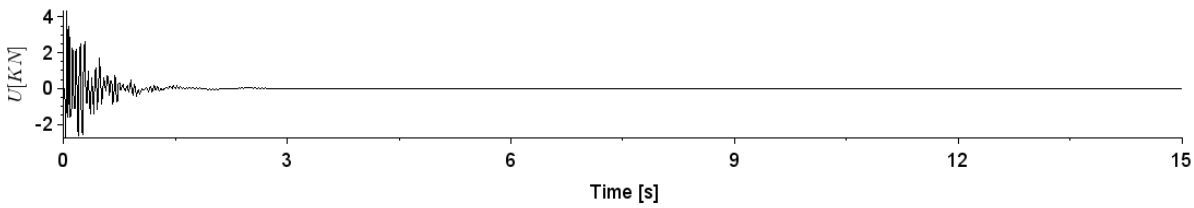

Figure 4.2: Boundary control $U(t)$.

The displacement of the beam without control (i.e., with control gains $k = 0$) is shown on the left side of Figure 4.1. From the results presented, it can be observed that there are large vibrations along the beam. In the right side of Figure 4.1, the beam displacement after activating the proposed control ($k = 50$) is shown. As can be seen, the control law successfully reduces the vibrations and stabilizes the system. The temporal behavior of the boundary control $U(t)$ is shown in the Figure 4.2.

5 Conclusion

This work succeeded in proposing an ideal linear control of the Euler-Bernoulli beam and we were able to stabilize the beam energy exponentially under free vibrations.

Conflict of interest

The authors have no conflicts of interest to declare.

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