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Strong stability in a queueing system with unknown general distribution

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Résumé In this paper, we discuss the approximation issue of queueing systems governed at least by a general unknown distribution G, using the strong stability method. Since the proximity of the characteristics of the real and ideal systems depends on the density function of the unknown general law G, it may be first estimated using the kernel density method taking into account boundary effects. Numerical applications are provided to support the theoretical results.

Keywords : Queue, Approximation, Strong stability, Kernel density, Boundary effects.

8.1 Introduction

The aim of this work is to show the applicability of the strong stability method [1, 2, 7] to precise the proximity of the characteristics of two systems, when at least one of the distributions governing one of the systems is general and unknown so its density function must be estimated by using the kernel density estimation method [9, 8].

The strong stability method states that the perturbation done must be small, in the sense that the general law G of arrivals (resp. service times) must be close but not equal to the Poisson (resp. exponential) one. Consequently, the density function of the law G must be close to the density function of the exponential law which is defined on a bounded support $[0, \infty[$. Thus, the boundary effects must be taken in consideration when using the kernel density method [10, 6].

8.2 Kernel density estimation method

Let $X_1, ..., X_n$ be a sample coming from a random variable X of density function f and distribution F. The Parzen-Rosenblatt kernel estimate [9, 8] of the density $f(x)$ for each point $x \in \mathbb{R}$ is given by :

$$f_n(x) = \frac{1}{nh_n} \sum_{j=1}^{n} K(\frac{x - X_j}{h_n}) \tag{8.1}$$

K is a symmetric density function called kernel and h_n is called bandwidth.

8.2.1 Boundary effects

Several results are known in the literature when the density function is defined on the real line $\mathbb{R}$ [8, 9]. In the case of a density function defined on a bounded support, the boundary effects are present. To resolve this problem, many recent methods have been elaborated [10, 6].

Schuster estimator ("mirror image")

Schuster [10] suggests to create the mirror image of the data in the other side of the boundary and then apply the estimator (8.1) for the set of the initial data and their reflection. $f(x)$ is then estimated, for $x \geq 0$, as follows :

$$\tilde{f}_n(x) = \frac{1}{nh_n}\sum_{j=1}^{n}[K(\frac{x-X_j}{h_n}) + K(\frac{x+X_j}{h_n})] \tag{8.2}$$

Asymmetric kernel estimators

A simple idea to avoid boundary effects, is the use of a flexible kernel, which never assign a weight out of the support of the density function and which correct automatically and implicitly the boundary effects. We can cite the asymmetric kernels [6] given by :

$$\hat{f}_b(x) = \frac{1}{n}\sum_{i=1}^{n} K(x,b)(X_i), \tag{8.3}$$

where b is the bandwidth and the asymmetric kernel K can be taken as a Gamma density K_G with parameters $(x/b+1, b)$, given by :

$$K_G(\frac{x}{b}+1, b)(t) = \frac{t^{x/b}e^{-t/b}}{b^{x/b+1}\Gamma(x/b+1)}, \tag{8.4}$$

8.3 Strong stability in $M/M/1$ system after perturbation of arrival flow

Théorème 8.1 *[4]*
Suppose that the charge ($\frac{\lambda}{\gamma}$) of the $M/M/1$ system is smaller than 1. Therefore, for all β such that $1 < \beta < \frac{\gamma}{\lambda}$, the imbedded Markov chain X_n is v-strongly stable, after a small perturbation of the inter-arrival times, for $v(k) = \beta^k$, and $\|P^* - P\|_v \leq (1+\beta)w$.
In addition, if $w < \frac{(1-\rho)(\gamma-\lambda\beta)}{(1+\beta)(2\gamma-\lambda(1+\beta))}$, we have :

$$Err = \|\pi^* - \pi\|_\upsilon \le \frac{(1+\beta)(2\gamma - \lambda(1+\beta))(\gamma-\lambda)w}{\frac{(\beta-1)(\gamma-\lambda\beta)^3}{(\beta-1)\gamma+\lambda\beta} - (2\gamma-\lambda(1+\beta))(1+\beta)(\gamma-\lambda\beta)w} \tag{8.5}$$

$$w = w(H, E_\lambda) = \int_0^\infty |H - E_\lambda|(dt) \tag{8.6}$$

$$\begin{cases} \pi_k^* = \lim_{n\to\infty} Pr(X_n^* = k), k = 0,1,2,..., \\ \pi_k = \lim_{n\to\infty} Pr(X_n^= k), k = 0,1,2,..., \end{cases} \tag{8.7}$$

$\rho = \beta \frac{\lambda}{\gamma - \frac{\gamma}{\beta} + \lambda}$.

8.4 Strong stability in $M/M/1$ system after perturbation of service times

Théorème 8.2 *[5]*
Suppose that the charge $\frac{\lambda}{\mu}$ of the $M/M/1$ system is smaller than 1. Then, for all β such that $1 < \beta < \frac{\mu}{\lambda}$, the imbedded Markov chain X_n is υ-strongly stable, after a small perturbation of the service times, for $\upsilon(k) = \beta^k$.
Suppose that $\int_0^{+\infty} t|F - E_\mu|(dt) \le \frac{w'}{\lambda}$, then : $\|P - P'\|_\upsilon \le \beta_0 w'$.
In addition, if $w' \le \frac{(1-\rho)}{C\beta_0}$, we have :

$$Err' = \|\pi - \pi'\|_\upsilon \le \beta_0 w' C C' (1 - \rho - C\beta_0 w')^{-1} \tag{8.8}$$

$$w' = w'(F, E_\mu) = \int_0^{+\infty} |F - E_\mu|(dt) \tag{8.9}$$

$\rho = \frac{\mu}{\beta(\mu+\lambda-\beta\lambda)} < 1$, $C' = \frac{\mu-\lambda}{\mu-\lambda\beta}$, $C = \frac{2\mu-\lambda(1+\beta)}{\mu-\lambda\beta}$.
$\beta_0 = \max\left\{\beta : 1 < \beta < \frac{\mu}{\lambda} \text{ and } \int_0^{+\infty} e^{\lambda(\beta-1)t}|F - E_\mu(t)|(dt) < \beta w'\right\}$

8.5 Kernel density method for the approximation of the $G/M/1$ system by the $M/M/1$ system using the strong stability method

We want to apply the kernel density method to estimate numerically the proximity of the $G/M/1$ and $M/M/1$ systems, by evaluating the variation distance w defined in (8.6) and the error Err defined in (8.5) between the stationary distributions of the tow according systems when applying the strong stability method.
We apply the same steps of the algorithm given in [4, 5]. The programming is done with Matlab 6.5 environment.

8.5.1 Simulation study 1

First case : Consider a $G/M/1$ system such that the density function of the general law G is given by :

$$g(x) = \begin{cases} \frac{1}{2}e^{-x} + e^{-2x}, & \text{if } x \geq 0\,; \\ 0, & \text{otherwise.} \end{cases} \tag{8.10}$$

By generating a sample coming from the general law G with the density function $g(x)$ defined in (8.10), we use the kernel density method to estimate this latter by using the different estimators given in the three following cases :
Second case : We use the density function $g_n(x)$ found by applying the Parzen-Rosenblatt estimator defined in (8.1).
Third case : We use the density function $\tilde{g}_n(x)$ found by applying the Schuster estimator defined in (8.2).
Fourth case : We use the density function $\hat{g}_n(x)$ found by applying the asymmetric kernel estimator defined in (8.3) with the Gamma kernel given in (8.4).
For the last three cases, we take : the sample size $n = 200$, the number of simulations $R = 100$. In all the cases, we introduce the service mean rate $\gamma = 10$.
We obtain the results in Table 8.1 :

Table 8.1. w and Err measures for different estimators

	$g(x)$	$g_n(x)$	$\tilde{g}_n(x)$	$\hat{g}_b(x)$
Variation distance w	0.0711	0.2104	0.0985	0.0820
Error Err	0.21		0.35	0.26

8.6 Kernel density method for the approximation of the $M/G/1$ system by the $M/M/1$ system using the strong stability method

We want to apply the kernel density method to estimate numerically the proximity of the $M/G/1$ and $M/M/1$ systems, by evaluating the variation distance w' defined in (8.9) and the error Err' defined in (8.8) between the stationary distributions of the tow according systems when applying the strong stability method.

8.6.1 Simulation study 2

We follow the same process used in the simulation study 1.
Consider an $M/Cox2/1$ system such that the law of inter-arrivals is assumed to be exponential with parameter λ and the service law is effectuated in two steps, it's density function is given by :

$$g(x) = \begin{cases} (1-a)\mu_1 e^{-\mu_1 x} + \frac{a\mu_2}{\mu_2-\mu_1}\mu_1 e^{-\mu_1 x} + \frac{a\mu_1}{\mu_1-\mu_2}\mu_2 e^{-\mu_2 x}, & \text{if } x \geq 0 \text{ and } \mu_1 \neq \mu_2\,; \\ (1-a)\mu_1 e^{-\mu_1 x} + a\mu_1 e^{-\mu_1 x}(\mu_1 x), & \text{if } x \geq 0 \text{ and } \mu_1 = \mu_2\,; \\ 0, & \text{otherwise.} \end{cases}$$

We take the parameters of the law Cox2 : $\mu_1 = 3$, $\mu_2 = 10$, $a = 0.005$. In all the cases, we introduce the inter-arrivals mean rate : $\lambda = 0.5$. We obtain the results in Table 8.2 :

Table 8.2. w' and Err' measures for different estimators

	$g(x)$	$g_n(x)$	$\tilde{g}_n(x)$	$\hat{g}_b(x)$
Service mean rate μ	2.9955	2.9690	2.9846	2.9937
Charge of the system $\frac{\lambda}{\mu}$	0.1669	0.1684	0.1675	0.1670
stability domain $(1 < \beta < \frac{\mu}{\lambda})$	$]1, 5.99105[$	$]1, 5.9379[$	$]1, 5.9692[$	$]1, 5.9874[$
Variation distance w'	0.0012	0.2361	0.0289	0.0097
Error Err'	0.0435		0.0713	0.0587

8.7 Perspectives

1. For the same work done in [3], apply the algorithm in [4, 5] to determine W^* and the error on the stationary distributions, with no restriction of conditions.
2. Apply it again with the different estimators of kernel density method.
3. Apply the discrete event simulation and the Student test to determine the proximity of the characteristics of the two systems using the different estimators.

8.8 conclusion

In this paper, we show the applicability of the strong stability method to evaluate the proximity of the characteristics of tow systems when at least one of the distributions governing one of the systems is general and unknown so its density function must be estimated by the means of the kernel density estimation method.

We show the interest of taking into consideration the correction of the boundary effects when using the kernel density method for the study of the strong stability of a queueing system with general unknown distribution, in order to substitute the characteristics of a complicated real system by those of a simpler classical system.

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