Nonconvex Quadratic Minimization with One Negative Eigenvalue

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Abstract

This project provides a new support method of global optimization to solve the quadratic minimization problem with one negative eigenvalue, subject to box constraints. We investigate the support of the objective function and exploit properties of the indefinite associated matrix for finding global optimality criterion (necessary and sufficient conditions). Furthermore, using these conditions and computational techniques, we apply the support method that can effectively solve a quadratic minimization problem with an indefinite associated matrix, having one negative eigenvalue. Particularly, we study the case where the associated matrix is positive subdefinite, and we use the suggested support algorithm in order to find the optimal solution. We present numerical applications to solve some box-constrained nonconvex problems with one negative eigenvalue.

Introduction

The resolution of a quadratic problem with linear constraints is very difficult in the nonconvex case, clearly the nonconvex quadratic problems are NP-Complete. In particular, global quadratic minimization problem with one negative eigenvalue is NP-hard [2]. So the global research of the solutions is a very difficult and very complicated application, and several efforts have been made to find efficient methods in order to simplify the resolution of this type of problems[1].
Our contribution in this paper is the development of a new method for solving the nonconvex quadratic problem, where the associated matrix is indefinite and contains precisely one negative eigenvalue. In particular, the problem with an associated positive subdefinite matrix is often not NP-hard [3].

Model description

We consider the nonconvex quadratic minimization problem with box constraints:

$$(QP) \quad \min F(x) = \frac{1}{2}x^t Dx + c^t x, \tag{1}$$

$$s.t \quad \ell_i \leq x_i \leq u_i, \qquad i = \overline{1,n}, \tag{2}$$

where $D^t = D = (d_{ij}, 1 \leq i,j \leq n)$ is a symmetric matrix of order n, supposed indefinite with one negative eigenvalue (in particular, D is positive subdefinite).

Definition

The real symmetric matrix D is called Merely Positive SubDefinite matrices (MPSubD : matrices that are not PSD), if and only if
1. $\eta(\mathrm{D}) = 1$,
2. $D \leq 0 \Leftrightarrow D = (d_{ij} \leq 0, 1 \leq i,j \leq n)$ and $D \neq 0$,
where $\eta(\mathrm{D})$ is the number of the negative eigenvalues of D.

First order local optimality necessary conditions

Let x be a global (local) minimum of (QP). Then the following conditions must be satisfied:

$$\begin{aligned} &E_i(x) \geq 0, \quad \forall\, i \in J_L = \{i \in J: \ x_i = \ell_i\},\\ &E_i(x) \leq 0, \quad \forall\, i \in J_U = \{i \in J: \ x_i = u_i\},\\ &E_i(x) = 0, \quad \forall\, i \in J_F = \{i \in J: \ \ell_i < x_i < u_i\}, \end{aligned} \tag{3}$$

where $E = Dx + c$ is the gradient of the objective function F at x.

Second order local optimality necessary conditions

Let x be a stationary point of the problem (QP). Then the following condition

$$D_F = D(J_F, J_F) \succcurlyeq 0 \ \ (J_F \ is\ defined\ in\ (3)\ and\ verifies\ E(J_F) = 0) \tag{4}$$

is necessary for the global (local) optimality of the vector x.

Second order optimality sufficient conditions

Let x be a stationary point verifying the conditions (3) and we consider the set

$$J_0 = \{i \in J: \ E_i = 0\}. \tag{5}$$

If $D(J_0, J_0) \succ 0$, then x is a local minimum of the problem (QP).

Global optimality criterion

Given:

$$F(\overline{x}) - F(x) = E^t(x)\Delta x + \frac{1}{2}\Delta x^t Q \Delta x + \frac{1}{2}\Delta x^t (D - Q)\Delta x,$$

where the matrix $Q = diag(\alpha_1, ..., \alpha_n)$, $\alpha_i \in \mathbb{R}$, is constructed such that $D - Q \succcurlyeq 0$, with D supposed MPSubD or indefinite having one negative eigenvalue.
We can generate the matrix Q as follows
a) $Q_1 = \overline{D}$, where $\overline{D} = diag(\overline{d}_1, ..., \overline{d}_n)$, $\overline{d}_i \in \mathbb{R}$, is constructed such that $D - \overline{D} \succcurlyeq 0$. So we define $\overline{d}_i$ as follows:

$$\overline{d}_i = d_{ii} - \sum_{j=1, j\neq i}^{n} |d_{ij}|, \ \ \forall i = 1, ..., n.$$

The matrix $(D - \overline{D})$ will be diagonally dominant with nonnegative diagonal elements. Hence we deduce that $D - \overline{D} \succcurlyeq 0$.

b) $Q_2 = \lambda_1 I_n$, where λ_1 is the negative eigenvalue of the matrix D, and I_n is an identity matrix of order n. Consequently, we get $D - \lambda_1 I_n \succcurlyeq 0$.

It is preferable to construct another matrix Q combining the matrices Q_1 and Q_2 [5]. So, in order to satisfy the global optimality criterion, we chose an arbitrary real number $\rho \in [0,1]$ and we determine Q as follows: $Q = \rho Q_1 + (1-\rho)Q_2 = diag(\alpha_1, ..., \alpha_n)$, where $\alpha_i = \rho \overline{d}_i + (1-\rho)\lambda_1$, $i = 1, ..., n$. Now, we define the matrix $\widehat{Q} = diag(\widehat{\alpha}_1, ..., \widehat{\alpha}_n)$, where the numbers $\widehat{\alpha}_i$, $i = 1, ..., n$ are defined as follows:

$$\widehat{\alpha}_i = \min\{0, \alpha_i\} = \begin{cases} \alpha_i, \ if \ \ \alpha_i < 0, \\ 0, \ \ \ if \ \ \alpha_i \geq 0, \ i \in J. \end{cases} \tag{6}$$

Sufficient optimality conditions

Theorem:
Let x be a feasible solution (FS) of the problem (QP) and we note by $\widehat{E}$ the vector of estimations such that

$$\widehat{E}_i(x) = \begin{cases} E_i(x) + \frac{1}{2}\widehat{\alpha}_i(u_i - \ell_i), \ \ if \ \ x_i = \ell_i, \\ E_i(x) + \frac{1}{2}\widehat{\alpha}_i(\ell_i - u_i), \ \ if \ \ x_i = u_i, \\ E_i^2(x) - \frac{1}{2}\widehat{\alpha}_i(u_i - \ell_i), \ if \ \ \ell_i < x_i < u_i, \ i \in J. \end{cases} \tag{7}$$

Then the following conditions:

$$\begin{cases} \widehat{E}_i(x) \geq 0, \ if \ \ x_i = \ell_i, \\ \widehat{E}_i(x) \leq 0, \ if \ \ x_i = u_i, \\ \widehat{E}_i(x) = 0, \ if \ \ \ell_i < x_i < u_i, \ i \in J, \end{cases} \tag{8}$$

are sufficient for the global optimality of the vector x.

Results

Example

Consider a problem of quadratic minimization with one negative eigenvalue given as follows :

$$(QP) \quad MinF(x) = -x_1^2 - x_2^2 - 2x_1x_2 + x_1$$

$$s.t \quad -2 \leq x_i \leq 2, \qquad i = 1, 2$$

where :

$$D = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}.$$

Such D is a positive subdefinite matrix having one negative eigenvalue: $\lambda_1 = -4$, let's notice that this problem is concave. This example has two local minima: $x_*^1 = (-2,-2)$ and $x_*^2 = (2,2)$; furthermore the vector $x_*^1 = (-2,-2)$ verifies the sufficient global optimality conditions, then it is the global minimum of (QP) with $F(x_*^1) = -18$ as it is shown in picture1 with Matlab:

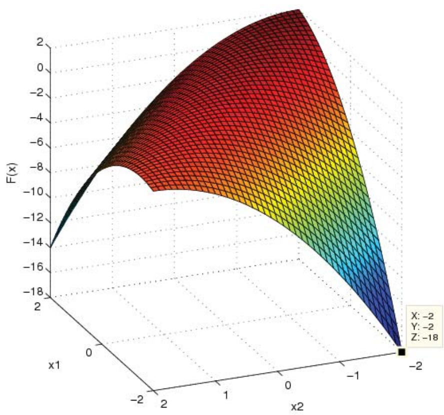

Figure 1: Example of Quadratic minimization with One Negative Eigenvalue.

Conclusion

We have considered an indefinite quadratic problem with box constraints, where the corresponding matrix has one negative eigenvalue. In particular, when the matrix D is merely positive subdefinite, we have proved that the global minimum is an extreme point. We have developed a new support method for solving the nonconvex problems while investigating the support of the objective function. We have presented the algorithm which can find a global minimum, while starting by an initial Support Feasible Solution. So, if the global optimality criterion is verified, then the SFS is optimal, else we generate an other SFS.

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