Insensitive Bounds of M/G/1 Retrial Queue with Two-way Communication and Multiples types of Outgoing Calls

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Introduction

Using approximation methods is essential to deal with the complexity of communication protocols especially in a call center context, where the server not only serves incoming calls, but in idle time it makes outgoing calls to the outside. This type of systems with both incoming and outgoing calls is referred to as a two-way communication retrial queueing model [1]. In this work, we use the stochastic comparison approach to investigate the monotonicity properties of an M/G/1 retrial queue with two way communication and multiples types of outgoing calls. we propose to bound it by a new Markov chain which is easier to solve by using stochastic comparison approach drawing on results achieved by Boualem et al. [2]

Mathematical model description

The model considered in this work comes from Sakurai and Phung-Duc (2015) [3], where a single server retrial queues with two-way communication and multiples of outgoing calls is considered.
Primary incoming calls arrive according to a Poisson process with rate λ. if the server is idle at arriving of incoming call, it will be served immediately. Else, incoming call joins the orbit and repeats its request after an exponentially distributed time with rate n. If the server is free it makes an outgoing call of type "ℓ" in an exponentially distributed time with rate α_ℓ (ℓ =2…n+1). We assume that the two types of calls receive different service times. Indeed, the service times of incoming calls and an outgoing call of type "ɩ" are characterized by the distribution functions $S_1(x)$ and $S_\ell(x)$ (ℓ = 2…n+1) respectively.
Let C(t) denote the state of the server at time t

$$C(t)=\begin{cases} 0; \text{ if the server is idle;} \\ 1; \text{ if it is busy with an incoming call} \\ 2; \text{ if an outgoing call of type } \ell \text{ is in service} \end{cases}$$

N(t) denote the number of incoming calls in the orbit at time t.

2.1. Embedded Markov Chain

The embedded Markov chain represent the number of customers in the orbit at the service completion epoch of either an incoming call or an outgoing call.

Methodology

Monotonicity properties of the embedded Markov Chain

the one-step transition probabilities of this Markov chain are given as follows:

$$\begin{cases} p_{i,i-1}= \dfrac{n\mu}{\lambda+\sum_{\lambda=2}^{n+1}\alpha_\lambda+n\mu}k_1^0, i\geq 1 \\ p_{i,j} = \dfrac{\lambda}{\lambda+\sum_{\lambda=2}^{n+1}\alpha_\lambda+n\mu}k_1^{j-1} + \dfrac{\sum_{\lambda=2}^{n+1}\alpha_\lambda}{\lambda+\sum_{\lambda=2}^{n+1}\alpha_\lambda+n\mu}k_\lambda^{j-1} \\ + \dfrac{n\mu}{\lambda+\sum_{\lambda=2}^{n+1}\alpha_\lambda+n\mu}k_1^{j-i+1}, 0\leq i\leq j; \\ p_{i,j} = 0, i-1\geq 2 \end{cases}$$

$$k_l^j = \int_0^\infty \frac{(\lambda x)^j}{j!}e^{-\lambda x}dS_l(x), l=1\ldots n+1, j\in Z_+.$$

Definition:

Let X and Y be two random variables with distribution functions F and G respectively.

1. X is said to be smaller than Y with respect to usual stochastic order (written $X\leq_{st} Y$ if $F(t)\geq G(t)$, for all real t,
2. X is less than Y in convex order (written $X\leq_v Y$) if and only if $\int_x^{+\infty}(1-F(t))dt \leq \int_x^{+\infty}(1-G(t)dt$
3. If X and Y are discret random variables taking values in N, with distribution $p_i=P(X=i)$ and $q_i=P(Y=i)$ respectively, then $X\leq_{st} Y$ ($X\leq_v Y$) if and only if, $\bar{p}_i = \sum_{j\geq i} p_j \leq \bar{q}_i = \sum_{j\geq i} q_j$ ($\bar{\bar{p}}_i = \sum_{j\geq i}\bar{p}_j \leq \bar{\bar{q}}_i = \sum_{j\geq i}\bar{q}_j$)

Let Σ_1 and Σ_2 be two M/G/1 retrial queues with two way communication and multiples types of outgoing calls with parameters:

$$\lambda^{(i)}, \mu^{(i)}, \alpha_\lambda^{(i)}, S_1^{(i)}, S_\lambda^{(i)}, \pi_n^{(i)}, k_\lambda^{(i)}$$

Lemma:
if $\lambda^{(1)}\leq_{so}\lambda^{(2)}, S_1^{(1)}\leq_{so} S_1^{(2)} and (S_2^{(1)}\ldots S_{n+1}^{(1)})\leq_{so}(S_2^{(2)}\ldots S_{n+1}^{(2)})$ then

$$\{k_n^{(1)}\}\leq_{so}\{k_n^{(2)}\}, so=(st,v)$$

Now to every distribution $p=(p_n)_{n\geq 0}$ the transition operator r of the embedded Markov chain associates a distribution $r_p=q=(q_m)_{m\geq 0}$ such that: $q_m=\sum_{n\geq 0}p_n p_{n,m}$

Theorem 1
Under the condition $\underset{2\leq\lambda\leq n+1}{Sup} S_\lambda \leq_{st} S_1$
The transition operator r is monotone with respect to the stochastic order $\leq_{st}$, i.e. for any two distribution $p^{(1)}$ and $p^{(2)}$,

$$p^{(1)}\leq_{st} p^{(2)} \Rightarrow rp^{(1)}\leq_{st} rp^{(2)}$$

Theorem 2
Under the condition $\underset{2\leq\lambda\leq n+1}{Inf} S_\lambda \geq_v S_1$
the transition operator r is monotone with respect to the convex order $\leq_v$, i.e. for any two distribution $p^{(1)}$ and $p^{(2)}$:

$$p^{(1)}\leq_v p^{(2)}\Rightarrow rp^{(1)}\leq_v rp^{(2)}$$

Results

Comparability conditions of $r^{(1)}$ and $r^{(2)}$ relatively to stochastic and convex orders

Theorem 3.
For two transition operators $r^{(1)}$ and $r^{(2)}$ of the embedded Markov chains added to each model Σ_1 and Σ_2 , we have
If $\lambda^{(1)}\leq\lambda^{(2)}, \mu^{(1)}\geq\mu^{(2)}, \sum_{\lambda=2}^{n+1}\alpha_\lambda^{(1)}\leq\sum_{\lambda=2}^{n+1}\alpha_\lambda^{(2)}, S_1^{(1)}\leq_{so} S_1^{(2)} and S_\lambda^{(1)}\leq_{so} S_\lambda^{(2)}$
then $r^{(1)}\leq_{so} r^{(2)}$ i.e. for any distribution p we have:

$$r^{(1)}p\leq_{so} r^{(2)}p, \qquad so=(st,v)$$

Proof. We must prove that: $\bar{p}_{nm}^{(1)}\leq\bar{p}_{nm}^{(2)} and \bar{\bar{p}}_{nm}^{(1)}\leq\bar{\bar{p}}_{nm}^{(2)}$

comparability conditions of stationary distributions of the number of customers in the orbit for two system M/G/1 retrial queue with two way communication and multiple type of outgoing call, with respect to stochastic and convex orders.

Theorem 4.
Let $\pi_n^{(1)}$ and $\pi_n^{(2)}$ be the stationary distributions of the number of customers in Σ_1 and Σ_2 .
If $\lambda^{(1)}\leq\lambda^{(2)}, \mu^{(1)}\geq\mu^{(2)}, \sum_{\lambda=2}^{n+1}\alpha_\lambda^{(1)}\leq\sum_{\lambda=2}^{n+1}\alpha_\lambda^{(2)}, S_1^{(1)}\leq_{so} S_1^{(2)} and S_\lambda^{(1)}\leq_{so} S_\lambda^{(2)}$
and $\underset{2\leq\lambda\leq n+1}{Sup} S_\lambda\leq_{st} S_1$(resp. $\underset{2\leq\lambda\leq n+1}{Inf} S_\lambda\geq_v S_1$
then, $\{\pi_n^{(1)}\}\leq_{so}\{\pi_n^{(2)}\}$, where $so=(\text{st ou v})$

Conclusions

In this work we showed the monotonicity of the transition operator of the embedded Markov chain relative to the strong stochastic ordering and convex ordering, as well as the comparability of two transition operators. Bounds are also obtained for the stationary distribution of the number of customers at departures epochs.

Références

1. Artalejo, J. R. and T. Phung-Duc. Single server retrial queues with two way communication. Applied Mathematical Modelling 2013; 37: 1811-1822.
2. Boualem. M, Cherfaoui.M, Djellab. N and Aissani. D. Ingalités stochastiques pour le modèle d'attente M/G/1/1 avec rappels. Afrika Matematika 2017; 28(5-6): 851-868.
3. Sakurai. H and Phung-Duc. T. Two-way communication retrial queues with multiple types of outgoing calls. Sociedad de Estadstica e Investi-gacin Operati