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Nouvelle borne de perturbation du système $M/M/1/N$ via le développement en série

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Résumé In this paper, we establish a framework for robust sensitivity analysis of queues. Our leading example is the finite capacity $M/M/1/N$ queue and we analyze the sensitivity of this model with respect to the assumption that interarrival times are exponential distributed.

Keywords : Series expansion, Stationary distribution, Deviation matrix, Strong stability, Queueing Systems.

7.1 Introduction

We are interested in the performance of a system when some of its parameters or characteristics are changed. The system as given is modeled as a Markov chain with transition matrix P and stationary distribution P. We assume that P is aperiodic and unichain. In this work, the transition matrix P is that of embedded Markov chain, Y_n, in an $M/M/1/N$ queue. What would be the effect on the stationary behavior of the queue if we changed the density function of the inter-arrival times. Let Q denote the transition matrix of the embedded Markov chain, Y'_n , modeling the perturbed system and assume that Y'_n has unique stationary distribution Q. The question about the effect of switching from P to Q on the stationary behavior is expressed by $P - Q$.

In this paper, we develop a framework for robust sensitivity analysis of the $M/M/1$ queue with respect to the interarrival time distribution. In other words, we bound the sensitivity of the stationary distribution of the queue length process of the $M/M/1/N$ queue with respect to the assumption that interarrival times are exponential. In order to do so, we will consider the queue length process embedded at service completions in the $M/M/1/N$ loss queue.

7.2 Preliminaries and Notations

In this analysis, we use the norm $\|.\|_v$, also called v-norm, where $v \in \mathbb{R}^S$ is such that $v(i) \geq 1$ for all $i \in S$. For a column vector $w \in \mathbb{R}^S$, the v-norm is given by

$$\|w\|_v = \sup_{i \in S} \frac{|w(i)|}{v(i)}$$

and for a row vector $u^\top \in \mathbb{R}^S$, the v-norm is given by

$$\|u\|_v = \sum_{i \in S} |u(i)| v(i).$$

As usual, we write distributions as row vectors and performance functions as column vectors.
For a matrix $A \in \mathbb{R}^{S \times S}$, the v-norm is given by :

$$\|A\|_v = \sup_{i \in S} \frac{\sum_{j \in S} |A|(i,j)\ v(j)}{v(i)},$$

where $|A|(i,j)$ denotes the $(i,j)th$ element of the matrix of absolute values of A.

7.3 Description of $M/M/1/N$ and $G/M/1/N$ models

Consider a $G/M/1/N$ system where inter-arrival times are independently distributed with general distribution $G(t)$ and service times are distributed with $E_\mu(t)$ (exponential with parameter μ). N denote the buffer capacity of the queue.
Let Y'_n be the number of customers left behind in the system by the nth departure. It's easy to prove that Y'_n forms a Markov chain with a transition matrix ($Q = (q_{i,j})_{i,j \in E}$) where $E = \{0,\ 1,\ \ldots,\ N\}$.
Consider also an $M/M/1/N$ system, which has Poisson arrivals with parameter λ and the same distribution of the service time as the previous system. It is known that Y_n (the number of customers left behind in the system by the nth departure) forms a Markov chain with a transition matrix ($P = (p_{i,j})_{i,j \in E}$).
Designate by π_Q and Π_Q the stationary distribution and the ergodic projector of Y'_n.
Designate also by π_P and Π_P the stationary distribution and the ergodic projector of Y_n.
We write D for the deviation matrix associated with P, where :

$$D_P = \sum_{m=0}^{\infty} (P^m - \Pi_P) = \sum_{m=0}^{\infty} (P - \Pi_P)^m - \Pi_P.$$

We denote the taboo kernel of the $M/M/1/N$ queue with taboo set $\{0\}$ by $(T = (T_{i,j})_{i,j\in S})$, i.e.,

$$T_{ij} = \begin{cases} p_{ij} \text{ if } i > 0 \text{ and } j \geq 0, \\ 0 \quad \text{if } i = 0 \text{ and } j \geq 0. \end{cases}$$

7.4 Serie Expansion for the $M/M/1/N$ system

Lemma 7.1 *Let Y_n the imbedded Markov chain of the $M/M/1/N$ system with transition matrix P then a finite number N exists such that*

$$\|P^n - \Pi_P\|_v \leq c\beta^n,$$

$\forall\ n \geq N$, where $c < \infty$ and $\beta < 1$.

Lemma 7.2 *Let Y_n the imbedded Markov chain of the $M/M/1/N$ system with transition matrix P and D_P the deviation matrix associated with P. Then D_P is finite*

Lemma 7.3 *Let Y_n the imbedded Markov chain of the $M/M/1/N$ system with transition matrix P and Y'_n the imbedded Markov chain of the $G/M/1/N$ system with transition matrix Q. Under condition (C) it holds that :*

$$\Pi_Q = \Pi_P \sum_{n=0}^{\infty} ((Q-P)D)^n.$$

7.5 Numerical example

Let us consider a $E_\alpha/M/1/N$ system where the density function of the inter-arrival times is Erlang-α

$$g(x) = \begin{cases} \lambda e^{-\lambda t} \dfrac{(\lambda t)^{\alpha-1}}{(\alpha-1)!} & \text{if } t \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The service times are distributed with $E_\mu(t)$ with parameter $\mu = 8$.
We consider also an $M/M/1/N$ system, which has Poisson arrivals with parameter $\lambda = 5$ and the same distribution of the service time as the $E_2/M/1/N$ system. We give $N = 6$.
Now, we apply the above algorithm to compute the stationary distributions of the $E_\alpha/M/1/N$ system π_Q.
For different values of α, we compute the stationary distributions of the $E_\alpha/M/1/N$ system

and we compare it by the true stationary distribution ν of the same system. The obtained results are presented in this table

$\alpha = 2$		$\alpha = 3$		$\alpha = 4$	
π_Q	ν	π_Q	ν	π_Q	ν
0.3720241	0.3004547	0.4113790	0.6321934	0.3470184	0.8036947
0.2344126	0.2222756	0.1919496	0.2331022	0.0742075	0.1577860
0.1515023	0.1643935	0.1207620	0.0859455	0.1051198	0.0309773
0.1004190	0.1214270	0.0914442	0.0316760	0.1319887	0.0060810
0.0678310	0.0891462	0.0734356	0.0116376	0.1308508	0.0011922
0.0456699	0.0635610	0.0600007	0.0041763	0.1152372	0.0002303
0.0281411	0.0387420	0.0510288	0.0012690	0.0955775	0.0000386

7.6 Establishing Norm Bounds

Let

$$\hat{D} = \sum_{n \geq 0} T^n (I - \Pi_P),$$

where finiteness of $\hat{D}$ follows from the fact that the deviation matrix exists for finite state-space models.

Théorème 7.1 *Let*

$$\eta = \frac{1 + \|\pi_P\|_v}{1 - \|T\|_v}.$$

If $||T||_v < 1$, then

$$||\hat{D}||_v \leq \eta,$$

and, if in addition, $\eta||Q - P|| < 1$, then

$$\|\pi_Q - \pi_P\|_v \leq \|\pi_P\|_v \frac{\eta||Q-P||_v}{1 - \eta||Q-P||_v}.$$

Théorème 7.2 *Suppose that $\rho = \lambda/\mu < 1$. For all β such that*

$$1 \leq \beta < \frac{\mu}{\lambda},$$

it holds that

$$||T||_v \leq \frac{\lambda\beta}{\lambda + \mu - \dfrac{\mu}{\beta}} \left(1 - \left(\frac{\mu}{\beta(\lambda+\mu)}\right)^N\right) < 1$$

and

$$\|\pi_P\|_v = \frac{(1-\rho)(1-(\rho\beta)^{N+1})}{(1-\rho^{N+1})(1-\rho\beta)}.$$

For the following, let

$$W = ||G - E_\lambda||_{tv} = \int_0^\infty |G - E_\lambda|(dt)$$

and observe that

$$W \leq ||G - E_\lambda||_v,$$

for $\beta \geq 1$.

Théorème 7.3 *If* $1 \leq \beta < \mu/\lambda$*, then, for* $\beta \geq 1$*, it holds*

$$\|(P - Q)\|_v \leq (1 + \beta)W.$$

7.7 Robust Sensitivity Analysis

Elaborating on the norm bounds provided in the previous section, we will establish in the next theorem the robust sensitivity bound.

Théorème 7.4 *If*

$$(1 + \beta)W \frac{1 + \|\pi_{E_\lambda}\|_v}{1 - \|T\|_v} < 1,$$

then it holds that :

$$\limsup_{\|G - E_\lambda\| \to 0} \frac{\|\pi_Q - \pi_P\|_v}{\|G - E_\lambda\|_v} \leq \inf_{\beta \geq 1} ||\pi_{E_\lambda}||_v (1 + \beta) \frac{1 + \|\pi_{E_\lambda}\|_v}{1 - \|T\|_v}.$$

Conclusion

In this paper, we have developed a framework for robust sensitivity estimates for the finite $M/M/1$ queue with respect to a perturbation of the interarrival time distribution. The extension of our framework to more complex queues and networks of queues is topic of further research.

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