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An optimal approximation of the characteristics of the $GI/M/1$ queue with two-stage service policy

M. CHERFAOUI[a], A. BARECHE[b], D. AÏSSANI[c] and S. ADJABI[d]

Unité de recherche LaMOS (Laboratoires de Modélisation et d'Optimisation des Systèmes)
Université de Bejaia, Bejaia 06000, Algérie
Tél. (213) 34 21 51 88,
[a]email : mouloudcherfaoui2013@gmail.com
[b] email : aicha_bareche@yahoo.fr
[c] email : lamos_bejaia@hotmail.com
[d] email : adjabi@hotmail.com

Résumé In this work, we consider an $GI/M/1$ system with two-stage service policy, having a rate service (μ_1, μ_2), with using the strong stability method we establish the approximation conditions for the stationary characteristics of this system by those of the standard $GI/M/1$ system. Under assumption that the approximation conditions are satisfied, we give the estimate of the deviation (stability inequalities) between the stationary distribution of the $GI/M/1$ system with two-stage service policy and those of the standard $GI/M/1$ system for three considered cases : the standard system has a service rate μ_1, the standard system has a service rate μ_2 and the standard system has a service rate μ^* minimizing the deviation. To calculate these deviations, the situation is modeled by a mathematical optimization problem that belongs to the minimization of a constrained nonlinear multi-variable function. Finally, numerical studies are performed to support the theoretical obtained results.

Keywords : G-queue ; Perturbation ; Strong stability ; Constrained nonlinear optimization.

3.1 Introduction

A queueing system with two stage-policy is such a system in which the server starts to serve with rate of μ_1 customers per unit time until the number of customers in the system reaches λ. At this moment, the service rate is change to that of μ_2 customers per unit time and this rate continues until the system is empty [4].
Note that for some practical situations modeled by queueing systems with N-stage service policy, the performance evaluation is complex or impossible. Therefore, we seek sometimes to evaluate the performances of the threshold policies or/and the optimal policy. On the other hand, if the cost due to switching service rate, in the hysteretic queues, is not negligible compared to the costs of processing and queues, it is known that the hysteretic policy is preferable to the threshold policy. For this, it is very important to define the domain within the switching cost is non-negligible. For this end, to illustrate how this domain is determined, and to enrich studies on the $GI/M/1$ system with two-stage service policy, we propose, to study the quality of the approximation of the stationary characteristics of

a $GI/M/1$ system with two-stage service policy by those of the system with threshold policies and by those of the system with the optimal policy when we use the strong stability method [1, 3].

3.2 Models Description and Assumptions

Consider a $GI/M/1$ $(FIFO, \infty)$ queue for which we adopt two-stage service policy. The customers arrive according to a renewal process with inter-arrival times following a distribution function G of mean $1/\lambda$. The server is initially idle. On an arrival of a customer, the server starts to serve μ_1 customers per unit time. Note that the service times are exponentially distributed. If the number of customers reaches N, then the server immediately changes his service rate to μ_2 customers per unit time and finishes the current busy period, otherwise he finishes the busy period with service rate μ_1. The same service policy is applied to the forthcoming customers. We assume that $\sigma_2 = \lambda/\mu_2 < 1$ for the stability of the queue.

Let $\{\tilde{X}_n, n \geq 1\}$ be the process of the number of customers in the system seen by the n^{th}. To obtain a global transition operator of $\{\tilde{X}_n, n \geq 1\}$ we must determinate, firstly, the probability θ, which represents the probability that the system serves with a service rate μ_1. For this, we define another process $\{\bar{X}(t), t \geq 0\}$ which has two states $\{E_1, E_2\}$, such as : E_1 : The system serves with a service rate μ_1 and E_2 : The system serves with a service rate μ_2 and its transition's graph can be represented as follows (Figure 3.1) : and

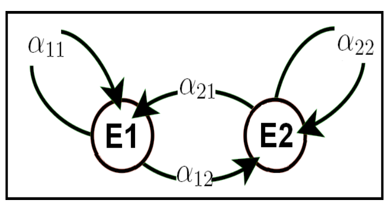

Figure 3.1. Transition's graph of the $\{\bar{X}(t), t \geq 0\}$ process.

its transition matrix α is given by : $\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$ where α_{ij} is the probability that the service rate of the process $\{\tilde{X}(t), T \geq 0\}$ passes from μ_i to μ_j $(i, j = 1, 2.)$. The α_{ij} is given by :

$\alpha_{12} = \pi^{(1)}(N-1) * P^*_{(N-1,N)}$, $\alpha_{11} = 1-\alpha_{12} = 1-\left(\pi^{(1)}(N-1) * P^*_{(N-1,N)}\right)$, $\alpha_{21} = \sum\limits_{i=1}^{\infty} \pi^{(2)}(i) * P^{(2)}_{(i,0)}$ and $\alpha_{22} = (1-\alpha_{21}) = 1 - \left(\sum\limits_{i=1}^{\infty} \pi^{(2)}(i) * P^{(2)}_{(i,0)}\right)$

where $P^*_{(N-1,N)} = \int\limits_0^\infty e^{-\mu_1 t} dG(t)$; and $\pi^{(1)}, \pi^{(2)}, P^{(1)}$ et $P^{(2)}$ are given by Kim et al. [4].
To determine the probability θ it is enough to study the stationary regime of the $\{\bar{X}(t), t \geq 0\}$ i.e.

$$(\theta, 1-\theta) \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} \theta \\ 1-\theta \end{pmatrix}$$

so, the solution of this system is :

$$\pi(E_1) = \theta = \frac{\alpha_{21}}{\alpha_{21} + \alpha_{12}} \quad and \quad \pi(E_2) = (1-\theta) = \bar{\theta} = \frac{\alpha_{12}}{\alpha_{21} + \alpha_{12}}.$$

Finally, we obtain the transition probabilities of $\{\tilde{X}_n, n \geq 1\}$ process which can be written

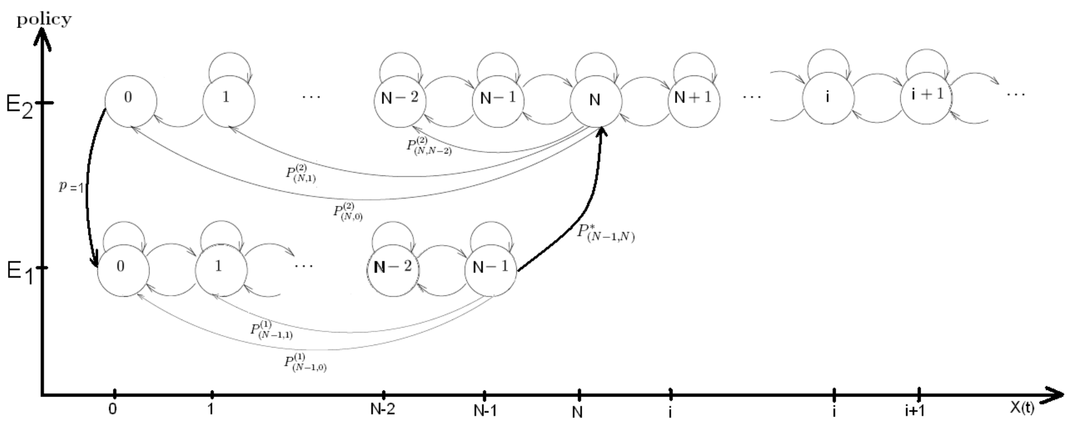

Figure 3.2. Transition's graph of the $\{\tilde{X}_n, n \geq 1\}$ process.

as follows :

$$\widetilde{P}_{ij} = \begin{cases} \int_0^\infty e^{-\mu_1 t} \, dG(t), & \text{if } i=0 \text{ and } j=1 \\ \int_0^\infty \left[\theta \, \frac{(\mu_1 t)^{i-j+1}}{(i-j+1)!} e^{-\mu_1 t} + \bar{\theta} \, \frac{(\mu_2 t)^{i-j+1}}{(i-j+1)!} e^{-\mu_2 t}\right] dG(t), & \text{if } 1 \leq j \leq i+1 \leq N \text{ and } i \geq 1\,; \\ \int_0^\infty \frac{(\mu_2 t)^{i-j+1}}{(i-j+1)!} \, e^{-\mu_2 t} dG(t), & \text{if } 1 \leq j \leq i+1 \text{ and } N < i+1\,; \\ 1 - \sum\limits_{j=1}^{i+1} \widetilde{P}_{ij}, & \text{if } j = 0\,; \\ 0, & \text{else.} \end{cases} \tag{3.1}$$

Consider also the standard $GI/M/1$ queueing system, the customers arrive according to a renewal process with inter-arrival times following a same distribution function G of mean $1/\lambda$ as the precedent queueing system. The service times are exponentially distributed with rate μ. Further, the embedded Markov chain $X = \{X_n, n \geq 1\}$, representing the number of customers in the $GI/M/1$ queueing system seen by the n^{th} arrival, has the following transition probabilities :

$$P_{ij} = \begin{cases} \int_0^\infty \frac{(\mu t)^{i-j+1}}{(i-j+1)!} e^{-\mu t} dG(t), \text{ if } 1 \leq j \leq i+1\,; \\ 1 - \sum_{j=1}^{i+1} P_{ij}, \quad \text{j=0}\,; \\ 0, \quad \text{else.} \end{cases} \tag{3.2}$$

Therefore, in the standard $GI/M/1$ queue, the steady-state probability distribution for the Markov chain, π_k $(k \in \mathbb{N})$, can be obtained by the usual procedure :

$$\pi_k = (1-\sigma)\sigma^k, k \geq 0, \tag{3.3}$$

where σ, $0 < \sigma < 1$, is the unique root of the equation :

$$x = G^*[\mu(1-x)]. \tag{3.4}$$

If $\lambda < \mu$, it can be shown that (3.4) has a unique solution within the interval $]0,1[$.

3.3 The strong stability method in the $GI/M/1$ queue with two-stage service policy

Let P and $\tilde{P}$ (respectively π and $\tilde{pi}$) denote the transition operators (respectively the stationary distributions) associated with the embedded Markov chains X and $\tilde{X}$ of the standard $GI/M/1$ queue with service rate μ and the $GI/M/1$ system with two-stage service policy with service rate $\mu 1$ and μ_2 and the switching level N.
The following theorem determines the strong -stability conditions of a $GI/M/1$ system after a small perturbation of the service policy. It also gives the estimate of the deviation of the stationary distributions.

Théorème 3.1 *[2] Suppose that in the $GI/M/1$ queueing system the geometric ergodicity condition, $\lambda/\mu < 1$, holds. Then, for all β such that $1 < \beta < 1/\sigma$, the embedded Markov chain $X = \{X_n, n \geq 1\}$ is υ-strongly stable for the test function $\upsilon(k) = \beta^k$. In addition, under the condition :*

$$\|\Delta\|_\upsilon < \frac{1-\rho}{c}, \tag{3.5}$$

we have :

$$\|\pi - \widetilde{\pi}\|_\upsilon \leq c_0 c \|\Delta\|_\upsilon (1 - \rho - c\|\Delta\|_\upsilon)^{-1} = E_\beta, \tag{3.6}$$

where $\|\Delta\|_\upsilon = \|P - \tilde{P}\|_\upsilon = \sup_{k\geq 0} \frac{1}{\beta^k} \sum_{j\geq 0} \beta^j |P_{kj} - \widetilde{P}_{kj}|$, $\rho = \beta \int_0^\infty e^{-[\mu(1-\frac{1}{\beta})]t} dG(t) = \beta G^*[\mu(1-\frac{1}{\beta})]$, $c_0 = \frac{1-\sigma}{1-\sigma\beta}$, $c = \frac{2-\sigma(\beta+1)}{1-\sigma\beta}$ and σ is the root of relation (3.4).

3.4 Approximation of the $GI/M/1$ queue with two-stage service policy

3.4.1 Optimal approximation

The determination of an optimal approximation consists to determine the parameters of the system with a single policy that minimizes the deviation between the characteristics of the $GI/M/1$ queue with two-stage service policy and those of the nominal $GI/M/1$ queue. Determining the parameters of the new system in our case is to solve the problem defined by 3.7 which belongs to the minimization of a constrained nonlinear multi-variable function. Indeed, the selection of the pair (β, μ) must minimize the deviation between the stationary probabilities, given by formula 3.6, of the two systems which is a nonlinear multi-variable function and verify the stability conditions (constraints) 3.5 and $1 < \beta < 1/\sigma$.

$$\begin{aligned} &\min_{(\beta,\mu)} \leftarrow c_0 c \|\Delta\|_v (1 - \rho - c\|\Delta\|_v)^{-1} \\ &\textit{such that} \begin{cases} \|\Delta\|_v c - 1 - \rho < 0 \\ 1 < \beta < 1/\sigma \\ \mu_1 \leq \mu \leq \mu_2 \end{cases} \end{aligned} \tag{3.7}$$

3.4.2 Approximation by the threshold policy

The objective of this section is to illustrate the manner in which we can verify the conditions and determine the error associated to the approximation of the characteristics of the $GI/M/1$ queue with two-stage service policy by those of the standard $GI/M/1$ having a service rate μ_1 (minimum policy) or μ_2 (maximum policy). In this case, we note that it sufficient to set the value of μ, in the various previous results, at μ_1 or μ_2, and solve the mathematical program 3.7, which becomes a constrained nonlinear optimization problem of a single-variable function. Indeed, the previous mathematical program will be written as follows :

$$\begin{aligned} &\min_{\beta} \leftarrow c_0^{(i)} c \|\Delta\|_v^{(i)} (1 - \rho^{(i)} - c\|\Delta\|_v^{(i)})^{-1} \\ &\textit{such that} \begin{cases} \|\Delta\|_v^{(i)} c^{(i)} - 1 - \rho < 0 \\ 1 < \beta < 1/\sigma^{(i)} \end{cases} \end{aligned} \tag{3.8}$$

3.5 Numerical Application

In this section, we present the solutions of mathematical models defined in formulas 3.7 and 3.8 applied on the $M/M/1$ system with two-stage service policy having an arrival rate $\lambda = 1$ and $N = 5$ as switching level of policy according to the service rate of each policy $\mu = (\mu_1, \mu_2)$ (see table 3.1).

μ_1	μ_2	Minimal threshold policy			Optimal policy			Maximal threshold policy		
		β^*	μ	Error	β^*	μ^*	Error	β^*	μ	Error
	5.04	1.1688		0.0529	1.1380	5.0038	0.0511	-		-
5	5.12	1.1695		0.1717	1.1385	5.0114	0.1657	-		-
	5.20	-		-	1.1497	5.0192	0.3005	-		-
	1.58	1.9060	μ_1	2.5669	2.0853	1.5042	1.0002	1.9060	μ_2	1.2691
1.5	1.62	2.0192		2.1542	2.1363	1.5108	1.1537	2.0135		1.5052
	1.70	2.2203		1.7554	2.3594	1.5179	1.0000	-		-
	1.14	1.2169		3.0594	1.3379	1.1381	1.0976	1.2186		1.1201
1.1	1.22	1.3585		2.3075	1.6379	1.1690	1.1037	-		-
	1.30	1.5053		2.0051	2.2983	1.1945	1.0432	1.2219		1.0525

TABLE 3.1: The different results of the model.

Discussion of the results : From the numerical results, obtained on the considered examples (see in the table 3.1), we note that :

- The approximation of the stationary characteristics of the $M/M/1$ system with two-stage service policy by those of standard $M/M/1$ system, having a service rate μ_1, have not always a sense. Indeed, for example, if $\mu_1 = 5$ and $\mu_2 = 5.16$ model 3.7 have not a solution. This means, the non-existence a value of the norm β satisfying the approximation conditions.
- Same observation, can be made on the approximation of the stationary characteristics of the $M/M/1$ system with two-stage service policy by those the standard $M/M/1$ system having a service rate μ_2.
- We can, always find a solution of the mathematical model 3.8, although the model 3.7 does not admit solution, that is to say, one can define a standard $M/M/1$ system having a service rate μ^* and its stationary characteristics are the best approximation of those $M/M/1$ system with two-stage service policy, having service rate (μ_1, μ_2), within the meaning of strong stability method.

3.6 Concluding Remarks

In this work, the modeling of the system $GI/M/1$ with two-stage service policy by a Markov chain with two states (E_1 : the system governed by a service rate μ_1 and E_2 : the system governed by a service rate μ_2) to determine the probability of θ (θ : the probability that the system is in the state E_1, $1 - \theta$ is the probability that the system is in the state

E_2), allowed us to calculate the global transition operator of the embedded Markov chain associated to the $GI/M/1$ system with two-stage service policy.
By exploiting the global transition operator and using the strong stability method, we have determined the approximation conditions of the stationary characteristics of the system $GI/M/1$ with two-stage service policy by those of the standard $GI/M/1$ system when it governs by one of the policy of the previous system and when it governs by an optimal policy and then we have given the deviation between the stationary distributions of each case.

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