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The bispectral representation of Markov switching BL models

A. GHEZAL

Département de Mathématiques, Université Constantine 1, Algérie.
email : `ahmedghezal25@yahoo.fr`

Résumé In this article, formulae for the third-order theoretical moments for superdiagonal and subdiagonal of the Markov-switching bilinear $\left(X_t = c\left(s_t\right) X_{t-k} e_{t-l} + e_t, k, l \in \mathbb{N}\right)$, and an expression for the bispectral density function are obtained.

Keywords : Markov-switching superdiagonal and subdiagonal Bilinear processes, Third-order moments, bispectral density function

13.1 Introduction

If the series is nonlinear the spectral will not adequately characterise the series. For instance, for some types of non linear time series (e.g. Markov switching bilinear models). As well, spectral analysis will not necessarily show up any features of non linearity (or non gaussianity) present in the series. It may be necessary, therefore, to perform higher order spectral analysis on the series in order to detect departures from linearity and Gaussianity. The simplest type of **bispectral analysis** notably by Rosenblatt and Van Ness (1965), Rosenblatt (1966), Van Ness (1966) and Brillinger and Rosenblatt (1967a, b).
Markov switching time series models (MSM) have received recently a growing interest because of their ability to adequately describe various observed time series subjected to changes in regime. An (MSM) is a discrete time random process $((X_t, s_t), t \in \mathbb{Z})$ such that (i) : $(s_t, t \in \mathbb{Z})$ is not observable, finite state, discret-time and homogeneous Markov chain and (ii) : the conditional distribution of X_k relative to its entire past, depends on (s_t) only through s_k. Flexibility is one of the main advantages of (MSM). The changes in regime can be smooth or abrupt, and they occur frequently or occasionally depending on the transition probability of the chain. Markov-switching models were introduced to the econometric mainstream by Hamilton [7, 8] and continue to gain popularity especially in financial time series analysis in order to integrated the mentioned characteristics in the conditional mean through local linearity representation. In this paper we alternatively propose a Markov switching bilinear $(MS-BL)$ representation, in which the process follows locally from

a bilinear characterization. This is in order to give a general, flexible and parsimonious framework for Markov switching modelling and $(MS - BL)$ have been extensively studied by Bibi, A., Aknouche, A. (2010). In this paper we shall consider a Markov-switching bilinear model defined by

$$X_t = c(s_t) X_{t-k} e_{t-l} + e_t,\ t \in \mathbb{Z}; \tag{13.1}$$

where $(e_t, t \in \mathbb{Z})$ is a strictly stationary and ergodic sequence of random variables with mean $E(e_t) = 0$ and variance $E(e_t^2) = 1$, for all t. The functions $a_i(s_t), b_j(s_t)$ and $c_{ij}(s_t)$ depend upon a time homogeneous Markov chain $(s_t, t \in \mathbb{Z})$ with finite state space $S = \{1; \ldots; d\}$, irreductible, aperiodic and ergodic, initial distribution $\pi(i) = P(s_1 = i), i = 1; \ldots; d$, $n-$step transition probabilities matrix $\mathbb{P}^n = \left(p_{ij}^{(n)}\right)_{(i,j)\in\mathbb{S}\times\mathbb{S}}$ where $p_{ij}^{(n)} = P(s_t = j\,|s_{t-n} = i)$ with $\mathbb{P} := (p_{ij})_{(i,j)\in\mathbb{S}\times\mathbb{S}}$ where $p_{ij} := p_{ij}^{(1)} = P(s_t = j\,|s_{t-1} = i)$ for $i; j \in \mathbb{S}$. In addition, we assume that e_t and $\{(X_{s-1}, s_t), s \le t\}$ are independent, we shall note

$$\mathbb{P}(M) = \begin{pmatrix} p_{11}M(1) \ldots p_{1d}M(1) \\ \vdots \quad \ldots \quad \vdots \\ p_{d1}M(d) \ldots p_{dd}M(d) \end{pmatrix}, \quad \mathit{\Pi}(M) = \begin{pmatrix} \pi(1)M(1) \\ \vdots \\ \pi(d)M(d) \end{pmatrix};$$

and $I_{(n)}$ is the $n \times n$ identity matrix. The model (13.1) is known as a superdiagonal model if $k > l$, and subdiagonal model for $k < l$. Let $(X_t, t \in \mathbb{Z})$ be a stationary time series satisfying the $MS - BL$ model (13.1), and the necessary condition for $(X_t, t \in \mathbb{Z})$ to be strictly stationary (see Bibi, A., Aknouche, A. (2010)). A sufficient condition for stationarity is $\gamma_L(A) < 0$, where $\gamma_L(A)$ is the Lyapunov exponent. The third-order moments of (X_t) are defined in [6] by :

$$\begin{aligned} R(r_1, r_2) &= E\{(X_t - \mu)(X_{t-r_1} - \mu)(X_{t-r_2} - \mu)\}; \\ &= E(X_t\ X_{t-r_1} X_{t-r_2}) - \mu(\gamma(r_1) + \gamma(r_2) + \gamma(r_1 - r_2)) + 2\mu^3; \end{aligned} \tag{13.2}$$

where $\mu = E(X_t)$, $\gamma(r) = E(X_t\ X_{t-r})$
It is sufficient to calculate $R(r_1, r_2)$ in the sector $0 \le r_1 \le r_2$ and the other values of $R(r_1, r_2)$ are determined from its symmetric relations (see Subba Rao and Gabr, (1984)). Lii and Rosenblatt (1982) have shown how bispectral density function can be used for estimating the phase relationships, and this in turn can be applied to the problem of deconvolution of e.g. seismic traces, quite a number of seismic records are observed to be non gaussian, and in many geophysical problems it is often required to estimate the coefficients. Also, the bispectral density function could, in principle be used for testing linearity.

The bispectrum has been used in a number of investigations as a data analytic tool ; we mention in particular the work of Hasselman, Munk and MacDonald (1963) on ocean waves, the papers of Lii, Rosenblatt (1979) on the energy transfer in grid generated turbulence. In this paper, we shall use the third-order moments to derive the bispectral density function of Markov switching $MS-BL$ models.

13.2 Spectral and bispectra

We now consider the evaluation of the spectral and bispectral of the process (X_t) when the process satisfies some linear time series models. We consider the linear model :

$$X_t = \sum_{j=0}^{q} b_j (s_t) e_{t-j}; \tag{13.3}$$

we have $E(X_t) = 0$, for all t,

$$\gamma(r) = E(X_t\ X_{t-r}) = \begin{cases} \sum\limits_{j=r}^{q} \underline{1}'_{(d)} \mathbb{P}\left(\underline{b}_j\right) \pi\left(\underline{b}_{j-r}\right) & \text{if } \ 0 \leq r \leq q; \\ 0 & \text{if } \ r > q. \end{cases}$$

The spectral density function $f(\omega)$ of the process (X_t) defined by $f(\omega) = \frac{1}{2\pi} \sum\limits_{r=-\infty}^{+\infty} \gamma(r) \exp(-ir\omega)$, $-\pi \leq \omega \leq \pi$, the spectral density function of the process (X_t) is given by $f(\omega) = \gamma(0) + 2 \sum\limits_{r=1}^{q} \gamma(r) \cos(\omega r)$, the bispectral density function $f(\omega_1, \omega_2)$ is given by $f(\omega_1, \omega_2) = 0$, all $\omega_1, \omega_2 \in [-\pi, \pi]$. We consider the linear model :

$$X_t = \sum_{i=1}^{p} a_i (s_t) X_{t-i} + \sum_{j=1}^{q} b_j (s_t) e_{t-j} + e_t. \tag{13.4}$$

Franq and Zakoïan (2001) propose the following representation of (13.4) :

$$\underline{X}_t = A(s_t) \underline{X}_{t-1} + \underline{e}_t,$$

where $\underline{X}_t = (X_t, X_{t-1}, ..., X_{t-p+1}, e_t, e_{t-1}, ..., e_{t-q+1})' \in \mathbb{R}^{p+q}$, $\underline{e}_t = (e_t, 0, ..., 0)' \in \mathbb{R}^{p+q}$ and

$$A(s_t) = \begin{bmatrix} a_1(s_t) & ... & a_p(s_t) & b_1(s_t) & ... & b_q(s_t) \\ 1 & 0 & ... & ... & ... & 0 \\ 0 & 1 & 0 & ... & ... & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & ... & ... & 0 & 1 & 0 \\ 0 & ... & ... & ... & ... & 0 \\ 0 & 1 & 0 & ... & ... & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & ... & ... & 0 & 1 & 0 \end{bmatrix}$$

$\underline{\gamma}(r)=E\left(\underline{X}_t\ \underline{X}'_{t-r}\right)$ is the autocovariance of $\underline{X}_t$. Then,

$$\pi(i)\,E\left(\underline{X}_t\ \underline{X}'_{t-r}\middle|\, s_t=i\right)=\sum_{j=1}^{d}A(i)\,E\left(\underline{X}_{t-1}\ \underline{X}'_{t-r}\middle|\, s_{t-1}=j\right)p_{ji}\pi(j);$$

for all $r>0$, we note :

$$\underline{W}(r)=\left(\pi(1)\,E\left(\underline{X}_t\ \underline{X}'_{t-r}\middle|\, s_t=1\right),...,\pi(d)\,E\left(\underline{X}_t\ \underline{X}'_{t-r}\middle|\, s_t=d\right)\right)'$$

(see Pataracchia. B (2011)) from which we have :

$$\underline{W}(r)=\mathbb{P}(\underline{A})\,\underline{W}(r-1)=\mathbb{P}^r(\underline{A})\,\underline{W}(0),\forall r>0,$$

where $\underline{A}=(A(1),...,A(d))'$. Finally, we can compute the autocovariance of the process $X_t:\gamma(r)=\left(\underline{H}'\otimes\underline{1}'_{(d)}\right)\underline{W}(r)\,\underline{H}$. For $r<0$, let us define :

$$\underline{\tilde{W}}(r)=\left(\pi(1)\,E\left(\underline{X}_t\ \underline{X}'_{t-r}\middle|\, s_{t-r}=1\right),...,\pi(d)\,E\left(\underline{X}_t\ \underline{X}'_{t-r}\middle|\, s_{t-r}=d\right)\right)'.$$

Then for $r<0$, $\underline{\tilde{W}}^{(i)}(r)=\pi(i)\,E\left(\underline{X}_t\ \underline{X}'_{t-r}\middle|\, s_{t-r}=i\right)=\left(\underline{W}^{(i)}(-r)\right)'$ from which we have $\underline{\tilde{W}}(r)=\underline{W}(-r)=\mathbb{P}^{-r}(\underline{A})\,\underline{W}(0),\forall r<0$. Finally, for negative r, we can compute the autocovariance of the process $X_t:\gamma(r)=\left(\underline{H}'\otimes\underline{1}'_{(d)}\right)\underline{\tilde{W}}(r)\,\underline{H}$, from which it can be verified that $\gamma(r)=\gamma(-r),\forall r<0$.

Spectral representation which defines the spectral as Fourier transform of the autocovariance function

$$\begin{aligned}
f(\omega)&=\frac{1}{2\pi}\sum_{r=-\infty}^{+\infty}\gamma(r)\exp(-ir\omega),\quad -\pi\leq\omega\leq\pi;\\
&=\frac{1}{2\pi}\left(\underline{H}'\otimes\underline{1}'_{(d)}\right)\sum_{r=-\infty}^{+\infty}\mathbb{P}^{|r|}(\underline{A})\exp(-ir\omega)\,\underline{W}(0)\,\underline{H};\\
&=\frac{1}{2\pi}\left(\underline{H}'\otimes\underline{1}'_{(d)}\right)\left(\mathbb{P}(\underline{A})-\mathbb{P}^{-1}(\underline{A})\right)\left(2\cos\omega\ I_{(d)}-\left(\mathbb{P}(\underline{A})+\mathbb{P}^{-1}(\underline{A})\right)\right)\underline{W}(0)\,\underline{H};
\end{aligned}$$

on conditional $\rho(\mathbb{P}(\underline{A}))<1$ (see Costa and all (2005)), the bispectral density function $f(\omega_1,\omega_2)$ is given by $f(\omega_1,\omega_2)=0$, all $\omega_1,\omega_2\in[-\pi,\pi]$. We consider the bilinear model :

$$X_t=\sum_{i=1}^{p}a_i(s_t)\,X_{t-i}+\sum_{j=1}^{q}b_j(s_t)\,e_{t-j}+\sum_{i,j=1}^{P,Q}c_{ij}(s_t)\,X_{t-i}e_{t-j}+e_t. \tag{13.5}$$

Bibi, A., Aknouche, A. (2010), propose the following representation of (13.5) $\underline{X}_t=B(s_t)\,\underline{X}_{t-1}+\underline{e}_t$, same result is obtained :

$$f(\omega)=\frac{1}{2\pi}\left(\underline{H}'\otimes\underline{1}'_{(d)}\right)\left(\mathbb{P}\left(\underline{B}\right)-\mathbb{P}^{-1}\left(\underline{B}\right)\right)\left(2\cos\omega\ I_{(d)}-\left(\mathbb{P}\left(\underline{B}\right)+\mathbb{P}^{-1}\left(\underline{B}\right)\right)\right)\underline{W}\left(0\right)\underline{H};$$

where $\underline{B}=\left(B\left(1\right),...,B\left(d\right)\right)'$. We note that sepectral representation does not allow us to distinguish linear models from nonlinear models and therefore should be talking about higher order spectral (bispectral).

13.3 Superdiagonal models

The superdiagonal model may be written as

$$X_t=c\left(s_t\right)X_{t-k}e_{t-k+m}+e_t,\ k\geq 2,\ 1\leq m\leq k-1; \tag{13.6}$$

we have $\mu=E\left(X_t\right)=0$, for all t,

$$\gamma\left(r\right)=E\left(X_t\ X_{t-r}\right)=\begin{cases}\underline{1}'_{(d)}\left(I_{(d)}-\mathbb{P}^k\left(\underline{c}^2\right)\right)^{-1}\underline{\pi}, & \text{if } r=0;\\ 0, & \text{if } r\neq 0.\end{cases}$$

Lemma 13.1 *For the* superdiagonal *model* (13.6) *all the third-order moments* $R\left(r_1,r_2\right)$ *are equal to zero except at* $r_1=k-m$, $r_2=k$, *viz.,* $R(k-m,k)=\underline{1}'_{(d)}\mathbb{P}^k\left(\underline{c}\right)\pi\left(\underline{V}\right)$, *where* $\pi\left(\underline{V}\right)=\left(\pi\left(1\right)E\left(X_t^2\left|s_t=1\right.\right),...,\pi\left(d\right)E\left(X_t^2\left|s_t=d\right.\right)\right)'$.

Proof : Consider the case $r_1=r_2=0$. Using (13.6) it can be shown that :

$$E\left(X_t^3\left|s_t=i\right.\right)=c^3\left(i\right)E\left(X_{t-k}^3e_{t-k+m}^3\left|s_t=i\right.\right)+3c\left(i\right)E\left(X_{t-k}e_{t-k+m}\left|s_t=i\right.\right)=0.$$

Using (13.2) we obtain, $R(0,0)=0$. For $r_1=r_2=r$, say, where $r>0$, we expand X_t using (13.3) to give

$$E\left(X_tX_{t-r}^2\left|s_t=i\right.\right)=c\left(i\right)E\left(X_{t-k}X_{t-r}^2e_{t-k+m}\left|s_t=i\right.\right)=0$$

Using (13.2) we obtain, $R(r,r)=0$. Now consider the case $r_1=0$ and $r_2=r$. Squaring both sides of (13.3), multiplying by X_{t-r} and taking expectations, we get :

$$E\left(X_t^2X_{t-r}\left|s_t=i\right.\right)=c^2\left(i\right)E\left(X_{t-k}^2X_{t-r}e_{t-k+m}^2\left|s_t=i\right.\right)=0;$$

we obtain, $R(0,r)=0$. Lastly, consider the case $r_1=r$ and $r_2=r+s$. When $r\geq 1$ and $s\geq 1$, it can be shown that

$$E\left(X_tX_{t-r}X_{t-r-s}\left|s_t=i\right.\right)=c\left(i\right)E\left(X_{t-k}X_{t-r}X_{t-r-s}e_{t-k+m}\left|s_t=i\right.\right);$$
$$E\left(X_tX_{t-r}X_{t-r-s}\left|s_t=i\right.\right)=\begin{cases}c\left(i\right)E\left(X_{t-k}^2\left|s_t=i\right.\right), & \text{if } r_1=k-m, r_2=k;\\ 0, & \text{otherwise.}\end{cases}$$

Using (13.2) we obtain, $R(k-m,k)=\underline{1}'_{(d)}\mathbb{P}^k\left(\underline{c}\right)\pi\left(\underline{V}\right)$.

13.4 Subdiagonal models

The subdiagonal model may be written as :

$$X_t = c(s_t) X_{t-1} e_{t-2} + e_t; \tag{13.7}$$

in which X_{t-1} and e_{t-2} are dependent, and therefore the derivation of the moments is more complicated and rather long. For this reason, we will present the final results. We have $\mu = E(X_t) = 0$, for all t,

$$var(X_t) = E\left(X_t^2\right) = \underline{1}'_{(d)} \left\{ \underline{\pi} + \left(I_{(d)} - \mathbb{P}\left(\underline{c}^2\right)\right)^{-1} \left(I_{(d)} + 2\mathbb{P}\left(\underline{c}^2\right)\right) \pi\left(\underline{c}^2\right) \right\},$$

and :

$$\gamma(r) = E(X_t\ X_{t-r}) = \begin{cases} \underline{1}'_{(d)} \mathbb{P}(\underline{c})\, \pi(\underline{c}), & \text{if } r = 3; \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, the third-order moments are given by

$$\begin{aligned} R(r_1, r_2) &= E(X_t\ X_{t-r_1} X_{t-r_2}) \\ &= \begin{cases} \underline{1}'_{(d)} \left\{ \pi(\underline{c}) + 3\left(I_{(d)} + 3\left(I_{(d)} - \mathbb{P}\left(\underline{c}^2\right)\right)^{-1} \mathbb{P}\left(\underline{c}^2\right)\right) \mathbb{P}(\underline{c})\, \pi\left(\underline{c}^2\right) \right\}, & \text{if } r_1 = 1,\ r_2 = 2 \\ 2\underline{1}'_{(d)} \mathbb{P}^2(\underline{c})\, \pi(\underline{c}), & \text{if } r_1 = 2,\ r_2 = 4 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

13.5 Bispectral structure

The bispectral density function is defined as :

$$f(\omega_1, \omega_2) = \frac{1}{4\pi^2} \sum_{r_1=-\infty}^{+\infty} \sum_{r_2=-\infty}^{+\infty} R(r_1, r_2) \exp(-ir_1\omega_1 - ir_2\omega_2);$$

where $R(r_1, r_2)$ is the third-order central moment defined by (13.2). Using the well known symmetric relations for both $R(r_1, r_2)$ and $f(\omega_1, \omega_2)$ (see, e.g., Subba Rao and Gabr, 1984) the bispectral density function $f(\omega_1, \omega_2)$ of the $MS-BL$ model (13.1) is given as follows. For the superdiagonal model (13.6) :

$$f(\omega_1, \omega_2) = \frac{R(k-m,k)}{4\pi^2} \left\{ \begin{array}{c} H(k-m,k) + H(k,k-m) + H(-m,-k) \\ +H(-k,-m) + H(m,-k+m) + H(-k+m,m) \end{array} \right\}; \tag{13.8}$$

where $H(r_1, r_2) = \exp(-ir_1\omega_1 - ir_2\omega_2)$. For the subdiagonal model (13.7), $f(\omega_1, \omega_2)$ given by :

$$f(\omega_1, \omega_2) = \frac{1}{4\pi^2} \left\{ \begin{array}{c} R(1;2) \left\{ \begin{array}{c} H(1;2) + H(2;1) + H(1;-1) + \\ H(-1;1) + H(-1,-2) + H(-2,-1) \end{array} \right\} \\ R(2;4) \left\{ \begin{array}{c} H(2;4) + H(4;2) + H(2;-2) + \\ H(-2;2) + H(-4,-2) + H(-2,-4) \end{array} \right\} \end{array} \right\}. \tag{13.9}$$

13.6 Conclusion

For the superdiagonal and subdiagonal bilinear models we have obtained all the theoretical third-order central moments and also explicit expressions for the bispectral density function. In practice, given real data $\{X_1, X_2, , ..., X_N\}$, both third-order moments and bispectral density function could be estimated (see, e.g., Subba Rao and Gabr, 1984).

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