Séminaire Mathématique de Béjaïa (LaMOS)
Volume 12, 2013, pp. 37 - 50

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A Closure Approximation for some Multi-servers queue using the strong stability method

M. CHERFAOUI[a], D. AïSSANI[b] and S. ADJABI[c]

Unité de recherche LaMOS (Laboratoires de Modélisation et d'Optimisation des Systèmes)
Université de Bejaia, Bejaia 06000, Algérie
Tél. (213) 34 21 51 88,
[a] email : mouloudcherfaoui2013@gmail.com
[b] email : lamos_bejaia@hotmail.com
[c] email : adjabi@hotmail.com

Résumé In this paper, we analyzed the stability of the $M/M/\infty$ system using the strong stability method, when this system is subject to a little perturbation at the level of the : • arrivals rate ($GI/M/\infty$), • structure ($GI/M/s$) and • service rate ($M/GI/\infty$).
For this purpose, we first determine the approximation conditions of the characteristics of the perturbed queuing system by those of the ideal system. Subsequently, under these conditions, we obtain the stability inequalities of the stationary distribution of the queue size.
Finally, to evaluate the performance of the strong stability method, we develop an algorithm that allows us to calculate the different theoretical results obtained and this in order to compare its output results with those of the simulation and to conclude on the quality of the method in question.

Keywords : Multi-servers queue ; Infinite-servers queue ; Embedded Markov chain ; Perturbation ; Strong stability ; Stability inequalities.

6.1 Introduction

The $\infty-$server queue and the 1-server queue are the most analytically tractable queues of practical importance. Thus, it makes sense to search for means of incorporating those available results into exact and approximation methods to study the stationary behavior of Markovian and non-Markovian $\infty-$server queue. That is, in fact, the main aim of this paper.
Since the analytical results obtained in the analysis of the multi-server system are only available in the form of a Laplace transform and/or a generating function, they cannot be used in practice due to their complexity. In this work, we are interested in the approximation of the characteristics of the $GI/M/\infty$ and $M/GI/\infty$ systems by those of the $M/M/\infty$ system and the characteristics of the $GI/M/s$ system by those of $GI/M/\infty$, in another word, we propose to study the stability of the system $M/M/\infty$ (as the ideal system) when the arrivals flow (respectively the service rate) is subject to small perturbations to obtain the $GI/M/\infty$ (respectively $M/GI/\infty$) and the stability of the system $GI/M/\infty$ (ideal

system) when the structure of this last system is subject to a perturbation at the number of servers level ($GI/M/s$).
However, in recent years, practical needs have driven the research towards the determination of estimates and quantitative performance measurement methods of stability. It is sometimes possible to estimate the numerically the error in the definition of the desired characteristics for small perturbations of parameters. For this, in this article, we propose to use the strong stability method because it performs, at the same time, a qualitative and quantitative analysis of the queuing systems.
Among the existing works in the literature carried out in the same direction as our proposal, on other waiting systems, we can quote the work of Benaouicha and Aïssani [5], Bouallouche and Aïssani [7, 8], Bareche and Aïssani [4], Berdjoudj et al. [6], ...
The first result is summarized in the construction of an upper bound of the absolute deviation (L_1 norm) between the stationary probabilities of the $M/M/s$ system and $M/M/\infty$ system, while the second result indicates that this absolute deviation tends to zero when the number of servers tends to infinity. The studies of Aïssani in 1992 [1, 2] where the stability of the system $GI/M/\infty$, by applying the strong stability method, was carried. In these last two works, the author had given the stability's conditions of the system with respect to the norm $v(k) = \beta^k(\beta > 1)$ and $k = 1, 2, ...$ and he proved that when s tends to the infinity, in the system $GI/M/s$, then the deviation, between transition operator of this system and the $GI/M/\infty$ system tends to zero. This leads the convergence of the deviation between their stationary probabilities to zero for the same norm. The work of Ramalhoto in 1999 [13], where the author gives a heuristic approximation of the infinite server queue by the multi-server queue with and without retrials.
The paper is organized as follows : in section 1 a brief description of the two systems $GI/M/\infty$ and GI/Ms are presented. In section 2, we present the preliminary concepts of strong stability method, then in section 3, we will the details of the study of the stability of $M/M/\infty$ and $GI/M/\infty$ systems. Finally, before concluding, we will give some numerical applications in section 4.

6.2 Description of $GI/M/\infty$ and $GI/M/s$ models

6.2.1 Description of $GI/M/\infty$ model

To analyze the $GI/M/\infty$ queue we can use the embedded markov chain technique which consists to identify a set of renewal points and relate the state probabilities at successive renewal points to each other.

For this, suppose that the customers arrive at epochs $T_1, T_2, ...$, and assume that the inter-arrival times $T_{k+1} - T_k$ $(k = 0, 1, ...;\ T_0 = 0)$ are random variables which are mutually independent and identically distributed $(i.i.d)$ with common distribution function $H(t) = \mathbb{P}\{T_{k+l} - T_k \le t\}(k = 0, 1, \ldots)$ and mean inter-arrival time $1/\lambda$. Let X_k be the number of customers in the system just prior the arrival of the kth customer ; X_k is the number of customers present at $T_k - 0$. Since the input is recurrent and the service times are by assumption $i.i.d$ exponential random variables with mean $1/\mu$ and independent of the arrival epochs, then the arrival epochs $T_1, T_2, ...$, are a renewal points. Hence, $X = (X_k; k = 0, 1, \ldots)$ is an homogeneous Markov chain with a state space $\mathbb{N} = \{0, 1, 2, \ldots\}$ and its transition probability :

$$P_{ij} = \mathbb{P}\{X_{k+1} = j/X_k = i;\ (j = 0, 1, \ldots; i = 0, 1, \ldots)\}; \tag{6.1}$$

are given as follows :

$$P_{ij} = \begin{cases} \int\limits_0^\infty C_{i+1}^j e^{-j\mu t} \left(1 - e^{-\mu t}\right)^{i+1-j} dH(t), \text{ if } i+1 \ge j\,; \\ 0, \qquad\qquad \text{else.} \end{cases} \tag{6.2}$$

It can be shown also, using the theory of Markov chains, that a unique stationary distribution

$$\pi_j = \lim_{k\to\infty} P\{X_k = j\};\ (j = 0, 1, \ldots); \tag{6.3}$$

exists if and only if $\int\limits_0^\infty t dH(t) < \infty$ [9].

Let consider the same situation as previous $(GI/M/\infty)$ but, this time we assume that the inter-arrival times $\tilde{T}_{k+1} - \tilde{T}_k(k = 0, 1, ...; T_0 = 0)$ are $i.i.d$ exponential random variables with mean $1/\lambda$. Let $\tilde{X}_k$ be the number of customers in the system just prior the arrival of the kth customer. The arrival epochs $\tilde{T}_1, \tilde{T}_2, \ldots$, in this case are also renewal points.

Therefore, $\tilde{X} = (\tilde{X}_k; k = 0, 1, \ldots)$ is an homogeneous Markov chain with a state space $\mathbb{N} = \{0, 1, 2, \ldots\}$. The transition probabilities

$$\tilde{P}_{ij} = P\{\tilde{X}_{k+1} = j/\tilde{X}_k = i\};\ (j = 0, 1, \ldots; i = 0, 1, \ldots); \tag{6.4}$$

are given as follows :

$$\tilde{P}_{ij} = \begin{cases} \int\limits_0^\infty C_{i+1}^j e^{-j\mu t} \left(1 - e^{-\mu t}\right)^{i+1-j} dE_\lambda, \text{ if } i+1 \ge j\,; \\ 0, \qquad\qquad \text{else.} \end{cases} \tag{6.5}$$

It can be shown, also using the theory of Markov chains, that the system has a unique stationary distribution defined as follow :

$$\tilde{\pi}_j = \lim_{k\to\infty} P\{\tilde{X}_k = j\} = \frac{(\lambda/\mu)^j}{j!} e^{-(\lambda/\mu)};\ (j = 0, 1, \ldots);\ (j = 0, 1, \ldots). \tag{6.6}$$

6.2.2 Description of $GI/M/s$ model

To analyze the $GI/M/s$ queue, we propose to use the same embedded Markov chain used in [9] and which consists to identify a set of renewal points, relate the state probabilities at successive renewal points to each other, assume the existence of a limiting stationary distribution, and solve the resulting system of equations.

Suppose that customers arrive at epochs $\hat{T}_1, \hat{T}_2, ...,$ and assume that the inter-arrival times $\hat{T}_{k+1} - \hat{T}_k (k = 0, 1, ...; \hat{T}_0 = 0)$ are random variables which are mutually independent and identically distributed $(i.i.d)$ with common distribution function $H(t) = \mathbb{P}\{\hat{T}_{k+l} - \hat{T}_k \leq t\}(k = 0, 1, \ldots)$ and mean inter-arrival time $1/\lambda$. All customers wait as long as necessary for service. Let $\hat{X}_k$ be the number of customers in the system just prior the arrival of the kth customer ; i.e. $\hat{X}_k$ is the number of customers present at $\hat{T}_k - 0$. Since the input is recurrent and the service times are by assumption $i.i.d$ exponential random variables with mean $1/\mu$ and independent of the arrival epochs, then the arrival epochs $\hat{T}_1, \hat{T}_2, ...,$ are renewal points. Hence, $\hat{X} = (\hat{X}_k; k = 0, 1, \ldots)$ is an homogeneous Markov chain with a state apace $\mathbb{N} = \{0, 1, 2, \ldots\}$. The transition probabilities

$$\hat{P}_{ij} = \mathbb{P}\{\hat{X}_{k+1} = j/\hat{X}_k = i\};\ (j = 0, 1, \ldots; i = 0, 1, \ldots); \tag{6.7}$$

of this system are given as follows,

$$\hat{P}_{ij} = \begin{cases} \int\limits_0^\infty \frac{(s\mu t)^{i+1-j}}{(i+1-j)!} e^{-s\mu t} dH(t), & \text{if } i \geq s-1, j \geq s \text{ and } i+1 \geq j; \\ \int\limits_0^\infty C_{i+1}^j e^{-j\mu t} \left(1 - e^{-\mu t}\right)^{i+1-j} dH(t), & \text{if } i \leq s-1 \text{ and } i+1 \geq j \\ \int\limits_0^\infty \int\limits_0^t C_s^j e^{-j\mu(t-\tau)} \left(1 - e^{-\mu(t-\tau)}\right)^{s-j} e^{-s\mu\tau} \frac{(s\mu\tau)}{(i-s)!}^{i-s} s\mu d\tau dH(t), & \text{if } i \geq s, j < s \text{ and } i+1 \geq j; \\ 0, & \text{else.} \end{cases} \tag{6.8}$$

It can be shown, using the theory of Markov chains, that a unique stationary distribution

$$\hat{\pi}_j = \lim_{k\to\infty} P\{\hat{X}_k = j\};\ (j = 0, 1, \ldots); \tag{6.9}$$

exists if and only if $\rho = \lambda/s\mu < 1$.

6.3 The strong stability criteria and preliminary notations

Let $\mathcal{M} = \{\nu_j\}$ be the space of finite measures on $\mathbb{N}$, and let $\mathcal{N} = \{f(j)\}$ be the space of bounded measurable functions on $\mathbb{N}$. We associate with each transition kernel P the linear mapping :

$$(\mu P)_k = \sum_{j\geq 0} \mu_j P_{ik}, \tag{6.10}$$

$$(Pf)(k) = \sum_{i \geq 0} f(i) P_{ki}. \tag{6.11}$$

Introduce on $\mathcal{M}$ the υ-norm of the form :

$$\|\nu\|_\upsilon = \sum_{j \geq 0} \upsilon(j) \mid \nu_j \mid, \tag{6.12}$$

where $\upsilon(k) = \beta^k$, for all $k \in \mathbb{N}$ and $\beta > 1$ is a real parameter. This norm induces in the space $\mathcal{N}$ the norm

$$\|f\|_\upsilon = \sup_{k \geq 0} \frac{|f(k)|}{\upsilon(k)}. \tag{6.13}$$

Moreover, for all $\nu \in \mathcal{M}$ and $f \in \mathcal{N}$, the symbols νf and $f \circ \nu$ denote respectively the summation and the kernel defined as below

$$\nu f = \sum_{k=0}^{+\infty} f(k) \nu_k, \tag{6.14}$$

$$(f \circ \mu)(k, j) = f(k)\, \mu_j, \text{ for all } (k, j) \in \mathbb{N} \times \mathbb{N}. \tag{6.15}$$

Let us consider $\mathcal{B}$, the space of linear operators, with the norm

$$\|Q\|_\upsilon = \sup_{k \geq 0} \frac{1}{\upsilon(k)} \sum_{j \geq 0} \upsilon(j) Q_{kj}. \tag{6.16}$$

Let ν and $\tilde{\nu}$ be two measures and suppose that these measures have finite υ-norm. For all f such that $|f(k)| \leq \Lambda \beta^k$ for some finite positive number Λ, we have

$$\begin{aligned} \mid \nu f - \tilde{\nu} f \mid &\leq \|\nu - \tilde{\nu}\|_\upsilon \|f\|_\upsilon \inf_{k \geq 0} \upsilon(k) \\ &= \|\nu - \tilde{\nu}\|_\upsilon \|f\|_\upsilon \\ &= \|\nu - \tilde{\nu}\|_\upsilon \sup_{k \geq 0} \tfrac{|f(k)|}{\beta^k}. \end{aligned} \tag{6.17}$$

Let us give the definition of the strong stability for an homogeneous Markov chain in the phase state $(\mathbb{N}, \mathcal{B}(\mathbb{N}))$ with respect to the υ-norm. Here $\mathcal{B}(\mathbb{N})$ is the σ-algebra generated by the singletons $\{j\}$.

Definition 6.1 *The Markov chain X with a transition kernel P and an invariant measure π is said to be strongly υ-stable with respect to the norm $\|.\|_\upsilon$ if $\|P\|_\upsilon < \infty$ and each stochastic kernel Q on the space $(\mathbb{N}, \mathcal{B}(\mathbb{N}))$ in some neighborhood*

$$\{Q : \|Q - P\| < \epsilon\}$$

has a unique invariant measure $\nu = \nu(Q)$ and $\|\pi - \nu\|_\upsilon \to 0$ as $\|Q - P\|_\upsilon \to 0$

The following theorem gives necessary and sufficient conditions for the strong stability of a Markov chain.

Théorème 6.1 ([3]) *A Markov chain X, with transition kernel P and stationary distribution π, is strongly υ-stable if and only if there exists a measure α and a nonnegative measurable function h on $\mathbb{E}$ such that*

a) $\pi h > 0,\ \alpha \boldsymbol{1} = 1,\ \alpha h > 0$ *;*

b) $\|P\|_\upsilon < \infty$ *;*

c) $T = P - h \circ \alpha \geq 0$ *;*

d) there exists $m \geq 1$ and $\rho < 1$ such that $T^m \upsilon(x) \leq \rho \upsilon(x)$ for all $x \in \mathbb{E}$;

where $\boldsymbol{1}$ is the identity function.

The quantitative estimates can be obtained by using the following results.

Théorème 6.2 ([11]) *Let X be a strongly $\upsilon-$stable Markov chain, with an invariant measure π and satisfying the conditions of theorem 6.1. If ν is the invariant measure of a kernel Q, then for the norm $\|Q - P\|_\upsilon$ sufficiently small, we have*

$$\nu = \pi \left[I - \Delta R_0 \left(I - \Pi\right)\right]^{-1} = \pi + \sum_{t \geq 1} \pi \left[\Delta R_0 \left(I - \Pi\right)\right]^t ;$$

where $\Delta = Q - P$, $R_0 = (I - T)^{-1}$, $\Pi = \boldsymbol{1} \circ \pi$ is the stationary projector of the kernel P and I the identity kernel on $\mathcal{M}$.

Corollary 6.1 *Under the conditions of theorem 6.1, for $\|\Delta\|_\upsilon < \frac{1-\rho}{c}$ we have the estimation :*

$$\|\mu - \pi\|_\upsilon \leq \|\Delta\|_\upsilon c \|\pi\|_\upsilon \left(1 - \rho - c\|\pi\|_\upsilon\right)^{-1},$$

where $c = m\|P\|_\upsilon^{m-1}\left(1 + \|\boldsymbol{1}\|_\upsilon \|\pi\|_\upsilon\right)$ and $\|\pi\|_\upsilon \leq (\alpha\upsilon)(1-\rho)^{-1}(\pi h) m \|P\|_\upsilon^{m-1}$.

6.4 Strong stability in the $M/M/\infty$ $(FCFS, \infty)$ system

6.4.1 Case : perturbation of the arrival rate

Strong stability conditions

the first step in implementing the strong stability method is determining the $\upsilon-$stability conditions of the considered system, in other word, it consists in delimiting the domain within the Markov chain $\tilde{X}_k$ associate to the analyzed system is strongly υ-stable after a small perturbation. In our case, the $\upsilon-$stability conditions of the $M/M/\infty$ system is given by the theorem 6.3.

Lemma 6.1 *Suppose that in the $M/M/\infty$ system, the condition $\lambda < \mu$ is fulfilled. Then, there exists $\beta \in \left]1, 1+\frac{(\mu-\lambda)(2\mu+\lambda)}{\lambda(\lambda+\mu)}\right[$ such that,*

$$\rho = \frac{1}{\beta}\left(\frac{\lambda(\beta-1)^2}{2\mu+\lambda} + \frac{2\lambda(\beta-1)}{\mu+\lambda} + 1\right) < 1. \tag{6.18}$$

Théorème 6.3 *Suppose that in the $M/M/\infty$ queueing system the condition of the lemma 6.1 holds. Then, for all β such that $1 < \beta < \beta_0$ the embedded Markov chain $\tilde{X}$ is $\upsilon-$strongly stable for the test function $\upsilon(k) = \beta^k$. Where :*

$$\beta_0 = \sup\{\beta/\beta > 1, \rho = \frac{1}{\beta}\left(\frac{\lambda(\beta-1)^2}{2\mu+\lambda} + \frac{2\lambda(\beta-1)}{\mu+\lambda} + 1\right) < 1.$$

Estimation of the strong stability

Before estimating the deviation between stationary distributions of the imbedded Markov chains $\tilde{X}$ and X using the strong stability method, we must estimate the deviation of transition operators firstly. This deviation, for the considered systems, is given by the following theorem.

Théorème 6.4 *Let $\tilde{P}$ (respectively P) be the transition operator of the embedded Markov chain in the $M/M/\infty$ system (respectively in the $GI/M/\infty$ system). Then, for all $1 < \beta < \beta_0$, we have :*

$$\|P - \widetilde{P}\|_{\upsilon} \leq \beta w, \tag{6.19}$$

where $w = \int\limits_0^\infty \mid H(t) - E_\lambda(t) \mid dt$.

After elaborating the stability conditions, it remains to determining the deviation between the stationary distributions of the imbedded Markov chains $\tilde{X}$ and X which can be done by using Theorem 6.2 and Corollary 6.1.

The following theorem allows us to obtain the stability inequalities with exactly computing of the constants.

Théorème 6.5 *Let $\tilde{\pi}$ and π be the stationary distributions of the embedded Markov chains $\tilde{X}$ and X respectively. Then, for all $1 < \beta < \beta_0$, and under the condition :*

$$w < \frac{(2\mu^2 - \lambda\mu\beta - \lambda^2\beta)(\beta-1)}{\beta^2(2\mu+\lambda)(\mu+\lambda)(1+e^{\lambda/\mu(\beta-1)})}, \tag{6.20}$$

we have :

$$\|\pi - \widetilde{\pi}\|_{\upsilon} \leq \frac{w\beta^2(2\mu+\lambda)(\mu+\lambda)\left(e^{\lambda/\mu(\beta-1)}\right)\left(1+e^{\lambda/\mu(\beta-1)}\right)}{(2\mu^2 - \lambda\mu\beta - \lambda^2\beta)(\beta-1) - w\beta^2(2\mu+\lambda)(\mu+\lambda)(1+e^{\lambda/\mu(\beta-1)})}; \tag{6.21}$$

where $w = \int\limits_0^\infty \mid H(t) - E_\lambda(t) \mid dt$.

6.4.2 Case : perturbation of the system structure

Strong stability conditions

In this case, the domain within the Markov chain $\tilde{X}_k$ associate to the analyzed system is strongly υ-stable after a small perturbation is given by the theorem 6.6.

Théorème 6.6 *Suppose that in the $GI/M/\infty$ system and suppose that the condition $\int_0^\infty dH(t)/t < \infty$ holds. Then, for all β such that $1 < \beta < \beta_0$, the embedded Markov chain $\tilde{X}$ is $\upsilon-$strongly stable for the test function $\upsilon(k) = \beta^k$. Where :* $\beta_0 = \sup\{\beta/\beta > 1, \rho = \frac{1}{\beta}\int\limits_0^\infty [1 - e^{-\mu t} + \beta e^{-\mu t}]^2 \, dH(t) < 1\}$.

Estimation of the strong stability

After elaborating the υ-stability conditions, it remains to determining the deviation between the stationary distributions of the imbedded Markov chains $\tilde{X}$ and $\hat{X}$ which can be done by using Theorem 6.2 and Corollary 6.1.
The following theorem allows us to obtain the stability inequalities with exactly computing of the constants.

Théorème 6.7 *Let $\tilde{\pi}$ and π be the stationary distributions of the embedded Markov chains $\tilde{X}$ and X, respectively. Then, for all $1 < \beta < \beta_0$, and under the condition :*

$$\|\varDelta\|_\upsilon < \frac{1-\rho}{c_0}, \tag{6.22}$$

we have :

$$\|\hat{\pi} - \widetilde{\pi}\|_\upsilon \leq c_0 c \|\varDelta\|_\upsilon (1 - \rho - c_0\|\varDelta\|_\upsilon)^{-1} = E_\beta; \tag{6.23}$$

where :

$$\begin{gathered} c = ||\widetilde{\pi}||_\upsilon = \sum_{n\geq 0} \prod_{k=1}^{n} \tfrac{g(k\mu)}{1-g(k\mu)} (\beta - 1)^n, \\ (here\ g(x) = \textstyle\int_0^\infty e^{-xt} dH(t)); \end{gathered} \tag{6.24}$$

$$c_0 = 1 + c; \tag{6.25}$$

and

$$\|\varDelta\|_\upsilon = \|\widetilde{P} - \hat{P}\|_\upsilon. \tag{6.26}$$

To illustrate the applicability of predicting performance perturbations, we will use the fact that our norm distance implies a bound on the effect of switching from the nominal chain to the perturbed one. For this, from the relation (6.17), we translate the norm bound in theorem 6.7 to bounds for individual performance measures f.

Corollary 6.2 *Let $\widetilde{\pi}$ and $\hat{\pi}$ be the stationary distributions of the imbedded Markov chains in the $GI/M/\infty$ system and $GI/M/s$ system, respectively. Suppose that the assumptions of Theorem 6.7 hold and $1 < \beta < \beta_0$, then, for any f such that $\|f\|_\upsilon < \infty$, it holds that*

$$|\pi f - \widetilde{\pi} f| \leq E_\beta \|f\|_\upsilon, \tag{6.27}$$

where E_β is defined in (6.23).

If we assume that the inter-arrival times are $i.i.d$ exponential random variables and we use the associated Markov chains to the $M/M/s$ and $M/M/\infty$ systems, then the Theorems 6.6 and 6.7 can be rewritten, respectively, as follow :

Théorème 6.8 *Suppose that in the $M/M/\infty$ queueing system the condition $\lambda/\mu < 1$ holds. Then, for all β such that $1 < \beta < \mu/\lambda$ the Markov chain $\tilde{X}$ is $\upsilon-$strongly stable for the test function $\upsilon(k) = \beta^k$.*

Théorème 6.9 *Let $\tilde{\pi}$ and π be the stationary distributions of the Markov chains $\tilde{X}$ and $\hat{X}$, respectively. Then, for all $1 < \beta < \mu/\lambda$, and under the condition :*

$$s \geq \alpha \left(\left(\frac{1+\beta^2}{\beta - 1} \right) \left(\frac{1+\alpha}{1-\alpha\beta} \right) \left(1 + e^{\alpha(\beta-1)}\right) - 1 \right); \tag{6.28}$$

we have

$$\|\pi - \tilde{\pi}\| \leq \frac{\alpha\,(1+\alpha)\,(1+\beta^2)\left(1+e^{\alpha(\beta-1)}\right)\left(e^{\alpha(\beta-1)}\right)}{(s+\alpha)\,(\beta-1)\,(1-\alpha\beta) - \alpha\,(1+\alpha)\,(1+\beta^2)\,(1+e^{\alpha(\beta-1)})}. \tag{6.29}$$

with $\alpha = \lambda/\mu$.

In addition, we can estimate the deviation between the two transition operators of the two systems in question, which is given by the following theorem.

Théorème 6.10 *Let $\tilde{P}$ (respectively $\hat{P}$) be the transition operator of the Markov chain in the $M/M/\infty$ system (respectively in the $M/M/s$ system). Then, for all $1 < \beta < \beta_0 = \mu/\lambda$, we have :*

$$\|P - \widetilde{P}\|_\upsilon = \frac{\alpha(1+\beta^2)}{\beta(s+\alpha)}, \text{ where } \alpha = \frac{\lambda}{\mu}. \tag{6.30}$$

Remark 6.1 *To prove the strong υ-stability of the imbedded Markov chain $\tilde{X}$ for the test function $\upsilon(k) = \beta^k$, where $\beta > 1$, we use the strong stability criterion (Theorem 6.1). For this, we choose the measurable function :*

$$h_i = \begin{cases} 0, \text{ if } i \geq 1\,; \\ 1, \text{ if } i = 0. \end{cases} \tag{6.31}$$

and

$$\alpha_j = P_{0j}(\infty) = \begin{cases} 0, & \text{if } j > 1\,; \\ \int_0^\infty (1 - e^{-\mu t})^{1-j} e^{-\mu j t} E_\lambda(t), & \text{if } j \leq 1\,; \end{cases} \tag{6.32}$$

for the first case and

$$\alpha_j = P_{0j}(\infty) = \begin{cases} 0, & \text{if } j > 1\,; \\ \int_0^\infty (1 - e^{-\mu t})^{1-j} e^{-\mu j t} dH(t), & \text{if } j \leq 1\,; \end{cases} \tag{6.33}$$

for the second case. Then, check conditions (a), (b), (c), and (d) of Theorem 6.1.

6.5 Numerical Application

In this section we present some applications examples of results obtained in previous sections, knowing that our goal is to validate and to illustrate the manner in which they can be exploited in practice.

To do this, we developed an algorithm that allows us to check the different conditions and calculate the various needed quantities where the steps of this algorithm is inspired from the algorithm proposed by Bouallouche-Medjkoune and Aïssani in [7, 8]. Thus, we obtain the following algorithm :

1. Introduce the parameters of the system (input).
2. Verify the existence of β_0
   if β_0 exists **then**
   the system is stable
   and **goto** 3
   else
   disp *'unable to conclude on the stability of the system'*
   and **goto** 4.
3. Determine the constant β_{opt} the value of β minimizing the bound (6.21) (respectively (6.23)) checking the constraint (6.20) (respectively (6.22))
4. end.

6.5.1 Case : perturbation of the arrival rate

The primary objective of this sub-section is to compare the bound put forward in Theorem 6.5 against that given by simulation. Therefore, we consider two examples $Cox_2/M/\infty$ and $E_2/M/\infty$ and we will apply our bounds put forward in Theorem 6.5 and simulation.

Example 1

In this example, we consider the $GI/M/\infty$ queuing system, where we set the service rate $\mu = 2$ and we assume that the distribution of the durations of inter-arrival $H(t)$ is a Coxian law with order two (a mixture of two exponential law), having the parameters $\lambda_1 = 1.25$, $\lambda_2 = 1.5$ and α and defined by its probability density $h(t)$ written as follow :

$$h(t) = \alpha\lambda_1 e^{-\lambda_1 t} + (1-\alpha)\lambda_2 e^{-\lambda_2 t}, \quad (t \geq 0,\ 0 \leq \alpha \leq 1). \tag{6.34}$$

The obtained results by the execution of the previous algorithm and by the discrete event simulation, for different values of $\alpha \in \{0.1, 0.2, ..., 0.9\}$, are ranked in table 6.1 below :

α	λ	β_{opt}	Algorithmic errors	Simulated errors
0.1	1.4706	1.2169	0.3923	0.1714
0.2	1.4423	1.2339	0.7611	0.1844
0.3	1.4151	1.2490	0.9987	0.3741
0.4	1.3889	1.2624	1.0476	0.9365
0.5	1.3636	1.2740	0.9387	0.7251
0.6	1.3393	1.2840	0.7461	0.5208
0.7	1.3158	1.2926	0.5317	0.3885
0.8	1.2931	1.2997	0.3292	0.2423
0.9	1.2712	1.3056	0.1513	0.1512

Table 6.1. Numeric results : Case $Cox_2/M/\infty$ ($\lambda = (1.25, 1.5)$ and $\mu = 2$).

Discussion of Results

We note that for small values of α, the deviation $\|\pi - \tilde{\pi}\|_v$ is small too, which is valid for large enough values of α. This can be explained by the fact that :

1)For small values of α the law $h(t) = \lambda_2 e^{-\lambda_2 t} + \epsilon$, which is very close to an exponential law with parameter λ_2.

2)For large values of α the law $h(t) = \lambda_1 e^{-\lambda_1 t} + \epsilon$, which is very close to an exponential law with parameter λ_1.

That is to say, when the value of α is large enough or small enough, the Coxian law tends to become an exponential law, hence the $Cox_2/M/\infty$ queue characteristics will be very close to those of the $M/M/\infty$ system.

If we compare the two numerical errors stored in the 4th and the 5th column of the Table 6.1 we see that the simulation results are always lower than the algorithmic results that warrants and confirms that bound E_β is an upper bound of the deviation $\|\pi - \tilde{\pi}\|_v$.

Example 2 :

In this example, we take the same position as the first example, except the law of inter-arrival $H(t)$ which will be an Erlang law of order two (the sum of two exponential) having the following probability density :

$$h(t) = \lambda^2 t e^{-\lambda t}, \quad t \geq 0. \tag{6.35}$$

λ/n	β_{opt}	Algorithmic errors	Simulated errors
0.1000	9.9781	0.4396	0.3375
0.1125	9.0923	0.5166	0.3722
0.1250	8.3852	0.6007	0.4008
0.1375	7.8081	0.6931	0.4086
0.1500	7.3285	0.7948	0.5435
0.1625	6.9241	0.9075	0.5482
0.1750	6.5787	1.0329	0.6868
0.1875	6.2806	1.1733	0.7029
0.2000	6.0210	1.3316	0.7824
> 0.2	-	-	-

Table 6.2. Numeric results : Case $E_2/M/\infty$.

The obtained results in this case, for different values of $\lambda \in \{0.200, 0.225, ...\}$, are ranked in Table 6.2.

Discussion of Results We note that, where the value of λ increases (arrivals rate λ/n increases) the difference $\|\pi - \tilde{\pi}\|_\upsilon$ increases too, and this until $\lambda = 0.4$ ($\lambda/n = 0.2$) which can be explained by the fact that Erlang law departs from the exponential law for a large enough values.

We also, note that for $\lambda \geq 0.425$ ($\lambda/n \geq 0.2125$) the stability conditions are not satisfied. This means that the system $M/M\infty$ is not $\nu-$strongly stable for the test function β^k, for this perturbation.

Comparing the two numerical errors stored in the 3th and the 4th column of the Table 6.2, we see that the bound E_β is an upper-bound of the deviation $\|\pi - \tilde{\pi}\|_\upsilon$ for all errors obtained by simulation which are below this bound.

6.5.2 Case : perturbation of the system service rate

In this sub-section we have consider the $M/M/\infty$ system subject to a little perturbation at the service rate level to obtain $Cox_2/M/\infty$ and $E_2/M/\infty$ queue. For the numerical application we set the inter-arrivals rate $\lambda = 1$ and we assume that the distribution of the durations of service time $H(t)$:

1st case : $h(t) = \alpha\mu_1 e^{-\mu_1 t} + (1-\alpha)\mu_2 e^{-\mu_2 t}$, with $t \geq 0$, $\mu = (1.25, 1.5)$, and $0 < \alpha < 1$) ;

2nd case $h(t) = \mu^2 t e^{-\mu t}, \quad t \geq 0$.

The obtained results by the execution of the previous algorithm and by the discrete event simulation, for different values of $\alpha \in \{0.1, 0.2, ..., 0.9\}$ for the Coxian law (respectively for $\mu = (0.200, 0.225, ...)$ for E_2 law), are presented in Figure 6.1 (respectively Figure 6.2).

Discussion of Results We note that the same conclusions and remarks than section 6.5.1 can be realized.

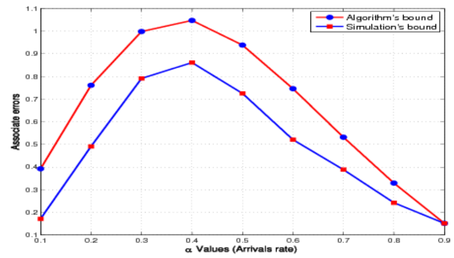

Figure 6.1. Comparative curves of the approximation errors : case Cox_2 law.

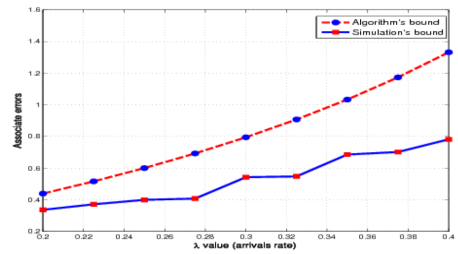

Figure 6.2. Comparative curves of the approximation errors : case E_2 law.

6.6 Conclusion

In this work, we applied for the first time the strong-stability method on the $M/M/\infty$ queue (respectively, $GI/M/\infty$ queue) which is subject, in a first case, to a small perturbation in the arrivals rate and in a second case, to a small perturbation in the service rate (respectively on its structure).

The application of the strong stability method, allows us to determine the stability's conditions of the $M/M/\infty$ system (respectively, $GI/M/\infty$ system) and a bounds of the stability inequalities between the stationary characteristics of $M/M/\infty$ system (respectively, $GI/M/\infty$ system) and those of $GI/M/\infty$ system in the case of the two perturbations considered.
To validate the obtained theoretical results, we have developed an algorithm that its role is to calculate with exactitude the stability inequalities constants. Finally, simulation studies validate the outputs of the algorithm execution on real examples.

Références

1. D. Aïssani. Strong stability of an embedded Markov chain in an $GI/M/\infty$ system. International Journal " Technologies Avancées", 2(1) : 33-38, 1992.
2. D. Aïssani. Estimation of the strong stability in an $GI/M/\infty$ system. International Journal " Technologies Avancées", 2(2) : 29-33, 1992.

3. D. Aïssani and N.V. Kartashov. Ergodicity and Stability of Markov Chains with Respect to Operator Topology in the Space of Transition Kernels. Compte Rendu Academy of Sciences U.S.S.R, (ser.A,11) :3-5, 1983.
4. A. Bareche and D. Aïssani. Kernel density in the study of the strong stability of the $M/M/1$ queueing system, Operations Research Letters,36 : 535–538, 2008.
5. M. Benaouicha And D. Aissani, Strong stability in a $GI/M/1$ queueing system, Theor. Probability and Math. Statist., 71,25–36, 2005.
6. L. Berdjoudj, M. Benaouicha and D. Aïssani. Measure of performances of the strong stability method. Mathematical and Computer Modelling, 56 : 241-246, 2012.
7. L. Bouallouche-Medjkoune and D. Aïssani, Measurement and performance of the strong stability method, Theor. Probab. Math. Stat. 72 (2005) :1–9, 2005.
8. L. Bouallouche-Medjkoune and D. Aïssani, Performance analysis approximation in a queueing system of type $M/G/1$, Math. Method Oper. Res. 63 (2) : 341–356, 2006.
9. R. B. Cooper's. Introduction to queueing theory. Computer systems and Management Science, Florida Atlantic University, Boca Raton, Florida, Second Edition. 1981.
10. J. M. Helary and R. Pedrono. Recherche opérationnelle, Travaux Dirigés. Hermann, 1983.
11. N. V. Kartashov, Strong stability of Markov chains. Journal of Soviet Mathematics, 34 : 1493–1498, 1986.
12. N. V. Kartashov, Strong Stable Markov Chains, VSP Utrech, TbiMC Scientific Publishers, Utrecht, Kiev, 1996.
13. M.F. Ramalhoto, The infinite server queue and heuristic approximations to the multi-server queue with and without retrials, Sociedad de Estadlstica e Investigacidn Operativa, 7(2), 333–350, 1999.