Séminaire Mathématique de Béjaïa (LaMOS)
Volume 12, 2013, pp. 11 - 13

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A Functional Approximation for Queues with Breakdowns

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Résumé In this paper we consider queues with breakdowns. We will develop a functional approximation of the stationary characteristics of this queue where the parameter of interest is the breakdown probability. More specifically, we will apply the strong stability method and the series expansion method. We provide an analysis of the $M/G/1$ queue with breakdowns for both finite and infinite waiting capacity.

Key words : Queues with break downs, functional approximation, strong stability method, series expansion method.

Server breakdowns are a common phenomenon in queueing networks. Unfortunately, the analysis of queueing systems becomes much more challenging through the occurrence of breakdowns. We assume that a server breaks down at the beginning of a service independently of everything else with probability θ, for $\theta \in [0,1]$. If a server breaks down it is repaired where we assume that repair times follow an exponential distribution. For our mathematical analysis we will elaborate on the Markov kernel of the queue length process embedded at appropriate events. For example, for the M/G/1 queue we will embed the Markov chain at service completions and repair completions. Let P_θ denote the Markov kernel of the (embedded) queue length process, then P_0 represents the system with no breakdowns whereas P_1 models the system with certain server breakdown. Due to our assumption that breakdowns occur independently of everything else, it holds that $P_\theta = \theta P_1 + (1-\theta)P_0$, for $\theta \in [0,1]$. For our analysis we consider the case that upon a server breakdown the customer is not lost but send back to the front of the queue. Note that this implies that P_1 will become a pure birth process as no customer will ever be served and the P_1 system is not stable, i.e., the mean queue length in the P_1 process will be unbounded. As we will discuss, a lower bound θ^* can be obtained such that for $\theta \leq \theta^*$ the mixed system P_θ is stable. Best to our knowledge this is a new approach to stability analysis.
Let π_θ denote the unique stationary distribution of P_θ, provided that it exists. Computing π_θ is a challenging problem and a variety of approaches have been proposed in the litera-

ture for approximately or indirectly solving the stationary distribution. The predominant approach is to obtain either the generating function of π_θ or an analytical expression for π_θ containing a Laplace-Stieltjes transform, see, for example, [1, 4]. Also numerical solutions by means of the matrix geometric method [17] are available, see [15, 18].

In performance analysis one is not only interested in evaluating the system for a certain set of parameters but also in the sensitivity of the performance with respect to the parameters. In a model with breakdowns, the breakdown probability is a parameter of key interest and we will analyze the dependence of π_θ on θ, which is a significantly more challenging then evaluating π_θ for fixed θ. An obvious approach would be to choose a sequence of reference points $\{\theta_n\}$, with, say, $a \leq \theta_n \leq b$, and use $\{\pi_{\theta_n}\}$ as approximation of π_θ for $a \leq \theta \leq b$. Unfortunately, numerical evaluation of π_θ at a sequence of points θ_n is computationally demanding. Moreover, if π_{θ_n} is obtained in only approximative form (by, say, evoking some numerical procedure), no information is available on the quality of the overall approximation of π_θ on $a \leq \theta \leq b$ by $\{\pi_{\theta_n}\}$. Rather than computing π_θ at various points independently, we will approximately compute π_θ through π_0, i.e., we will answer the question of what the effect of an increase of the breakdown probability by θ has on the system with no breakdowns. This kind of perturbation analysis is a classical research area in Markov chain theory, see, for example, [3, 7, 8, 11, 12, 14, 16, 19, 20, 21]. While these kind of bounds apply to queues with denumerable state space, the perturbation bounds provided in the literature for $\pi_\theta - \pi_0$ behave numerically rather poorly (we will illustrate this also by numerical examples).

For finite queues, the recently [9, 10] introduced series expansion algorithm (SEA) proved to be numerically efficient, however, it lacks applicability to denumerable queues. In addition, convergence of SEA hinges on checking algorithmically a contraction condition.

For our analysis we combine results from perturbation analysis of Markov chains with SEA. In particular, we will elaborate on the strong stability approach (SSA), [2, 5, 6, 13]. We discuss perturbation analysis of the M/M/c queue and the M/G/1 queue with breakdowns. The main findings we report are that, while the strong stability method has the advantage of providing bounds for infinite queues, unfortunately, the numerical quality of the bounds is rather poor. The series expansions proves to be numerically efficient but requires that a finite queue is studied. From this we conclude that the strong stability method is an analytical method that leads to qualitative bounds that can behave in practice rather poorly, whereas the series expansion method is a numerical approach that provides an efficient algorithm for the functional approximation of finite queues. There is, however, an interesting link between the two approaches. The techniques developed for the strong stability method lend themselves to establish lower bounds of convergence for series expansions and can be made fruitful for stability analysis. An interesting observation is that the strong stability methods can be efficiently applied to birth-and-death like queueing networks.

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