On Portfolio Selection when Asset Returns are Elliptical.

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Abstract

For many markets, returns have fatter than normal tails with an extremely high sample kurtosis. Hence the class of elliptical distributions provides attractive and appealing alternatives to normal distributions in modeling the empirical distribution of returns. We derive a result related to the covariance of functions of elliptical random variables. Based on this result, we obtain the conditions of optimality for the portfolio choice problem and show how to derive many useful results for the elliptical class of distributions including some main results in the studies of Chamberlain and of Owen and Rabinovitch. In particular, we obtain the efficient set, in the portfolio space, analytically. Furthermore, we derive a general global risk aversion measure, relevant to the case of elliptical risk, which generalizes the Rubinstein measure. We discuss, in the context of the portfolio choice problem, differences between normal and non normal elliptical distributions, especially those with heavier tails than normal, focusing on the role of Kurtosis as a measure of fatness of tails in determining the optimal investment strategy of risk-averse investors. If a riskless asset exists, the sensitivity of expected utility to kurtosis implies that a risk-averse investor demand for risk is smaller when faced with a fat tails elliptical distribution instead of a normal distribution that presents the same mean-variance choices. When the kurtosis measure gets sufficiently large, a risk-averse investor tends to invest all his wealth in the riskless asset even when the variance remains constant.

Finally, we show that short sales are not an optimal investment strategy for all risk-averse investors if and only if the means of asset returns are equal and the inverse of the variance-covariance matrix has non negative (positive) row sums.

Key words: Portfolio Selection; Elliptical Distribution; Scale Mixture of Normals; Kurtosis; Global Measure of Risk-Aversion; Stochastic Dominance.

Introduction

The theory of portfolio choice is central to most modern research in finance. It was clear since the early works of Knight (1921) and Hicks (1939) that the relevant parameters in a portfolio choice problem are return and risk. Until Morkowitz (1952) and Tobin (1958), those theoretical models, which did exist, failed to provide a useful measure of risk and to explain the diversification phenomenon.

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Morkowitz's theory begins with the simple assumption that investors consider only the mean and the variance of total returns on investments. Investors, according to the mean variance theory, dislike variance and seek to reduce it by diversifying given their subjective return to risk tradeoff.

Sharpe (1964), Lintner (1965) and Mossin (1966) built on Morkowitz's normative theory to develop a positive theory of market equilibrium. Their Capital Asset Pricing Model (CAPM) provides an equilibrium price of risk and implies that investors are compensated by a higher mean only for bearing systematic risk; all nonsystematic risk is diversified away. The extent of diversification in a particular portfolio is measured by the correlation coefficient with the market portfolio, which is the most diversified by definition.

If investors were Von Neuman-Morgenstern expected utility maximizers, then the mean-variance analysis could be justified by either assuming that utility functions were quadratic or that asset returns were normally distributed. Samuelson (1970) has shown that the mean-variance analysis is a good approximation for "compact" distributions and Merton (1971) has demonstrated its validity in intertemporal portfolio problems when trading takes place continuously.

Arrow (1971) objected to the use of quadratic utilities even in their monotone range. He observed that the quadratic function exhibits increasing absolute risk aversion, which implies that risky assets are inferior assets in a portfolio problem.

The objection to the normality of returns assumption was mainly on empirical grounds. It was noticed that the empirical distribution of asset returns has fatter tails than the normal in the sense that many observations are extreme and come from the tails of the distribution, which rejects normality. Mondelbrot (1963) and Fama (1965a) showed that stock return distributions are more peaked at the center with thicker tails than the normal. To capture the fat tail feature, they fitted the infinite variance stable Paretian class of distributions. Blattberg and Gonedes (1974) study indicates greater descriptive validity for the student t distribution over the stable Paretian as a model for stock market returns.

More recent studies confirmed the fat tail phenomenon. Longin (1995) used tests based on extreme values theory to study the behavior of daily returns in the French stock market over the period 1977-1990 and found deviation from normality with high kurtosis and relatively small negative skewness. His empirical results lead to the rejection of the normal and the discrete mixture of normal distributions which contain fewer extreme values than observed and the rejection of the stable Pareto-Levy which contains more extreme values. His results suggest the use of the student distribution. Duffie and Pan (1997) states that, for many markets, returns have fatter than normal tails. In particular, the S&P500 daily returns from 1986 to 1996 have an extremely high sample kurtosis and negative skewness.

The Indian stock market was studied by Broca (2002) for the period 1985-1998.The empirical series displays non normality with severe kurtosis and slight positive skewness. The results show that the student t model unquestionably provides a superior fit relative to the normal.

Fama and French (1995) found that non market factors such as size and the book to market ratio were priced by investors. Chung, Johnson and Schill (2001) examined returns for daily, weekly, monthly, quarterly and semi-annual intervals. They found that normality is rejected and reported higher than normal kurtosis for all five cases. They showed that adding systematic co-moments of order higher than two reduces the explanatory power of the Fama-French non market factors to insignificance in almost every case.
A general alternative to the mean-variance framework was developed by Hadar and Russel (1969) and Hanoch and Levy (1969). They have obtained first and second order stochastic dominance (FSD, SSD) which give optimal rules for the whole class of increasing utility functions and the class of risk-averse utilities respectively. Whitmore (1970) obtained third order stochastic dominance (TSD) with the additional restriction of the third derivatives of the utility function being positive. Bawa (1975) generalized TSD to the class of decreasing absolute risk-averse utilities. Unfortunately, this general framework proved to be too general to be very fruitful for portfolio analysis and asset pricing. Many authors generalized the results of the basic mean-variance model, in particular by studying the effect of higher moments on the portfolio choice problem and the pricing of risky assets. Fama (1965b) and Samuelson (1967) used the symmetric stable distribution for efficient portfolio selection. Rubinstein (1973) derived an equation for expected return in terms of an arbitrary number of co-moments. Kraus and Litzenberger (1976) developed a three moment CAPM for the valuation of risky assets. They argued that risk-averse investors prefer skewness. Owen and Rabinovitch (1983) discussed the relevance of the elliptical class of distributions to portfolio problems and showed that it extends the Tobin (1958) separation theorem, Bawa's (1975) rules of ordering uncertain prospects, Ross's (1978) mutual fund separation theorems and the results of CAPM to non normal symmetric distributions which are not necessarily stable. Chamberlain (1983) used the elliptical class to characterize asset returns distributions that imply mean-variance utilities. Simaan (1993a) developed a three parameter normative portfolio analysis where idiosyncratic security risks are modeled as following a joint elliptical distribution and skewness is generated by a single factor for the whole economy upon which different securities have different loadings. He showed that his generating process results in a three fund separation and provides a three parameter CAPM. Unlike Kraus and Litzenberger (1976) who used a truncated Taylor expansion of the utility function in their analysis, Simaan argued that the potential sensitivity of higher moments to skewness and the dependence of expected utility on such higher moments leave preference of risk-averse investors to skewness ambiguous.

Quite recently, Konno and Yamazaki (1991) proposed a linear programming model, as an alternative to the classical model of Morkowitz, which used mean absolute deviation (MAD) rather than variance as a measure risk. They showed that, if returns follow a multivariate normal distribution, then their model is equivalent to Morkowitz's. Speranza (1993) generalized the Konno and Yamazaki model using a risk function which is a linear combination of two semi absolute deviation from the mean. While Konno and Yamazaki (1991) showed that the MAD model did not require the covariance matrix, Simaan (1997) contends that the computational savings from the use of MAD objective are out weighted by the loss of information from the unused covariance matrix. He found that this would result in greater estimation risk.

The purpose of this article is to refocus attention on the portfolio choice problem using the framework suggested in Owen and Rabinovitch (1983) and Chamberlain (1983). In particular, we discuss in the context of the portfolio choice problem differences, if any, that may exist between the normal and a non normal elliptical distribution. Central to this concern is the role of Kurtosis as a measure of fatness of tails of the distribution of asset returns in determining the optimal strategy of risk-averse investors. This paper is organized as follows. In the next section, we present the class of elliptical distributions and discuss some of its properties that are relevant to portfolio analysis. In particular, we prove a lemma related to the covariance of functions of elliptical random variables which proved to be useful in determining the conditions of optimality in a portfolio choice problem. Section 3 is reserved to the analysis of portfolio selection problems when asset returns follow a multivariate elliptical distribution. We prove the main result in theorem 1 which enables us to determine, first, a general measure of global risk aversion for this class of distributions that generalize the Rubinstein (1976) measure. Secondly, the theorem is shown to provide a tool to establish many useful results for the elliptical class of distributions including some main results in Chamberlain (1983) and in Owen and Rabinovitch (1983). In portfolio selection context, we discuss differences between normal and non normal elliptical distributions, especially those with heavier tails than normal, focusing on the role of Kurtosis as a measure of fatness of tails in determining the optimal strategy of risk-averse investors. We conclude section 3 with a general diversification result for elliptical distributions. Finally, some concluding remarks are stated in the last section.

1. The Class of Elliptical Distributions

The class of elliptical distributions is examined in detail by Kelker (1970); it provides attractive and appealing alternatives to normal distributions and includes stable and non-stable members. Most members possess densities and when they exist their contours of constant probability are elliptical, hence the name. These densities, however, are more flexible than the normal density because their tails could be longer or shorter than that of the normal.

Let $\Delta \in R^n$ be a fixed vector and let $\underset{n\times n}{\Omega}$ be a positive definite symmetric matrix, in general, $X \in R^n$ is said to be elliptically distributed if and only if its characteristic function has the form:

$$C_X(t) = E\left(e^{it'X}\right) = \psi(t'\Omega t)e^{it'\Delta}$$

If we assume that $X \in R^n$ has a density then we have the following definition:

Definition: *$X \in R^n$ is said to be elliptically distributed with location vector Δ and characteristic matrix Ω if its density is of the form:*

$$f(X) = c_n |\Omega|^{-1/2} \phi\left((X-\Delta)' \Omega^{-1} (X-\Delta)\right)$$

The function $\phi(.)$ is a positive function of a scalar variable and c_n is a constant of integration.

When $X \in R^n$ has an elliptical distribution with location vector Δ and characteristic matrix Ω we write $X \sim \xi(\Delta, \Omega)$.

For example, if $X \in R^n$ is multivariate normal with mean vector Δ and variance covariance matrix Ω then its density is given by:

$f(X) = (2\pi)^{-n/2} |\Omega|^{-1/2} \exp\left(-\frac{1}{2}(X-\Delta)' \Omega^{-1} (X-\Delta)\right)$. Hence it is of the form given above.

Another example is the multivariate t-distribution with ν degree of freedom. Its density is given by:

$$f(X) = c_{n,\nu} |\Omega|^{-1/2} \left(1 + \frac{(X-\Delta)' \Omega^{-1} (X-\Delta)}{\nu}\right)^{\frac{-(\nu+n)}{2}}, \text{ where } c_{n,\nu} = \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\nu\pi)^{\frac{n}{2}}}$$

The class of multivariate elliptical distributions includes[2] the multivariate uniform distribution, the symmetric Kotz type distributions (including the normal distribution), the symmetric multivariate Pearson type II distribution, the symmetric multivariate Bessel distribution, the multivariate t-distribution, the symmetric logistic distribution and the symmetric multivariate stable distributions.

We list, without proofs[3], the following properties of elliptical distributions:

P1: Let $X \in R^n$ such that $X \sim \xi(\Delta, \Omega)$ and let $\underset{m \times n}{D}$ with rank $r(D) = m \leq n$ and define $Y = DX$ then $Y \sim \xi(D\Delta, D\Omega D')$.

P2: Let $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, $\Delta = \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix}$ and $\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}$

Then the conditional expectation is given by: $E(X_1 / X_2) = \Delta_1 + \Omega_{12}\Omega_{22}^{-1}(X_2 - \Delta_2)$

P3: Let $X \sim \xi(\Delta, \Omega)$ possesses k moments

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[2] See Gupta and Varga (1993), p.70, for definitions.

[3] [illegible]

If k>1 then $E(X)=\Delta$

If k>2 then the covariance matrix $\Sigma=\gamma\Omega$ for some $\gamma \in R_+$.

P4: Let $X \sim \xi(\Delta,\Omega)$, if any marginal density is normal then X has a normal distribution.

P5: Let $X \sim \xi(\Delta,\Omega)$, if Ω is a diagonal matrix, then the components of X are independent only if X has a normal distribution.

Before presenting the main results of this article we prove the following useful lemma which generalizes a known result for normal distributions. We assume, in what follows, that the density of the elliptical distribution exists and the second moments are finite.

Lemma: *Let $X \in R^2$ such that $X \sim \xi(\Delta,\Omega)$ with joint density $f(x_1,x_2)$ and marginal densities $f_1(x_1)$ and $f_2(x_2)$, we assume the finiteness of the first two moments where $E(X_i)=\Delta_i$, $\operatorname{cov}(X_i,X_j)=\sigma_{ij}$ $i,j=1,2$. Let $g()$ be a least once differentiable function.*

Define $h(x_2)=\int_{-\infty}^{x_2}\frac{\Delta_2-y}{\sigma_{22}}f_2(y)dy$ *and* $l(y)=\frac{h(y)}{f_2(y)}$

If $\lim_{y\to\pm\infty} g(y)h(y)=0$ *then* $\operatorname{cov}(X_1,g(X_2))=\sigma_{12}E(g'(X_2)l(X_2))$

Proof of the Lemma:

From the definition of the covariance we have:

$$\begin{aligned}
\operatorname{cov}(X_1,g(X_2)) &= E(X_1 g(X_2))-\Delta_1 E(g(X_2))\\
&= \iint_{x_1 x_2} x_1 g(x_2) f(x_1,x_2)dx_1dx_2-\Delta_1 E(g(X_2))\\
&= \int_{x_2} g(x_2)\left(\int_{x_1} x_1 f(x_1,x_2)dx_1\right)dx_2-\Delta_1 E(g(X_2))\\
&= \int_{x_2} g(x_2)\left(\int_{x_1} x_1 f(x_1/x_2)dx_1\right)f(x_2)dx_2-\Delta_1 E(g(X_2))\\
&= \int_{x_2} g(x_2)(E(X_1/x_2))f(x_2)dx_2-\Delta_1 E(g(X_2))
\end{aligned}$$

By P2: $E(X_1/x_2)=\Delta_1+\frac{\sigma_{12}}{\sigma_{22}}(x_2-\Delta_2)$

$$\therefore \operatorname{cov}(X_1, g(X_2)) = \int_{x_2} g(x_2)\left(\Delta_1 + \frac{\sigma_{12}}{\sigma_{22}}(x_2 - \Delta_2)\right) f(x_2)dx_2 - \Delta_1 E(g(X_2))$$

$$= \sigma_{12} \int_{x_2} g(x_2)\frac{x_2 - \Delta_2}{\sigma_{22}} f(x_2)dx_2$$

Letting $h(x_2) = \int_{-\infty}^{x_2} \frac{\Delta_2 - y}{\sigma_{22}} f_2(y)dy$ and integrating by parts, we get:

$$\operatorname{cov}(X_1, g(X_2)) = \sigma_{12}\left(-g(\infty)h(\infty) + g(-\infty)h(-\infty) + \int_{x_2} g'(x_2)h(x_2)dx_2\right)$$

Note that $h(-\infty) = h(\infty) = 0$ and define $l(y) = \frac{h(y)}{f_2(y)}$

If $\lim_{y \to \pm\infty} g(y)h(y) = 0$ then $\operatorname{cov}(X_1, g(X_2)) = \sigma_{12} E(g'(X_2)l(X_2))$ □

Remark: If $X \in R^2$ has the bivariate normal distribution then $l(y) = 1$ and $\operatorname{cov}(X_1, g(X_2)) = \sigma_{12} E(g'(X_2))$

The class of univariate elliptical distributions coincides[4] with the class of univariate distributions which are symmetric about a point (See Gupta and Varga 1993, p.70).

Let $X \in R$ be a univariate random variable with elliptical density $f(x) = f(x; \mu, \sigma^2)$ such that $E(X) = \mu$ and $V(X) = \sigma^2$ and define $h(x) = \int_{-\infty}^{x} \frac{\mu - y}{\sigma^2} f(y)dy$ and $l(x) = \frac{h(x)}{f(x)}$.

We would like to study here some properties of the functions $h(x)$ and $l(x)$. We show in the next proposition that if $f(x)$ is an elliptical density then $h(x)$ is an elliptical density and hence the function $l(x)$ can be interpreted as a likelihood ratio of two elliptical densities.

Proposition: *Let* $X \sim \xi(\mu, \sigma^2)$ *with density function* $f(x) = f(x; \mu, \sigma^2)$ *and define* $h(x) = \int_{-\infty}^{x} \frac{\mu - y}{\sigma^2} f(y)dy$ *then there exists an elliptical random variable* Y *whose*

[4] It should be clear that this class coincides also with the symmetric location scale parameters distributions in the sense of Bawa (1975). (See p.112 and p.116).

density is $h(x)$ *with* $E(Y)=E(X)=\mu$ *and* $V(Y)=\dfrac{E(X-\mu)^4}{3V(X)}$. *Furthermore,* $h(x)=f(x)$ *if and only if* $X\sim N(\mu,\sigma^2)$

Proof of the Proposition:

To prove the second part of the proposition it suffices to note that $X\sim N(\mu,\sigma^2)$ if and only if its density function $f(x)$ satisfies the differential equation $f'(x)=\dfrac{\mu-x}{\sigma^2}f(x)$.

As to the nature of the function $h(x)$, we notice first that $h(-\infty)=h(\infty)=0$. Secondly, since $\int_{\mu-x}^{\mu+x}\dfrac{\mu-y}{\sigma^2}f(y)dy=0$ and $f(x)$ is symmetric around μ, the function $h(x)$ is positive and symmetric around μ. It remains to show that $\int_{-\infty}^{+\infty}h(x)dx=1$.

Since $f(x)$ is the density of the elliptical random variable X then

$$f(x;\mu,\sigma^2)=\phi\left(\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

We have $h(x)=\int_{-\infty}^{x}\dfrac{\mu-y}{\sigma^2}f(y;\mu,\sigma^2)dy=\int_{-\infty}^{x}\dfrac{\mu-y}{\sigma^2}\phi\left(\left(\dfrac{x-\mu}{\sigma}\right)^2\right)dy$

Put $w=\dfrac{y-\mu}{\sigma}$ and write $h(x)=\int_{-\infty}^{\frac{x-\mu}{\sigma}}-w\phi(w^2)dw=\int_{\frac{\mu-x}{\sigma}}^{\infty}w\phi(w^2)dw$ but since

$\int_{-w_0}^{w_0}w\phi(w^2)dw=0$ we have $h(x)=\int_{\left|\frac{x-\mu}{\sigma}\right|}^{\infty}w\phi(w^2)dw$

But the random variable $W = \left|\frac{X-\mu}{\sigma}\right|$ has a probability density function[5] given by

$\varphi(w) = 2\sigma\phi(w^2)$ hence $h(x) = \frac{1}{2\sigma}\int_{\left|\frac{x-\mu}{\sigma}\right|}^{\infty} w\varphi(w)dw$

We have:

$$\int_{-\infty}^{\infty} h(x)dx = 2\int_{\mu}^{\infty} h(x)dx = \frac{1}{\sigma}\int_{\mu}^{\infty}\int_{\left|\frac{x-\mu}{\sigma}\right|}^{\infty} t\varphi(t)dtdx = \int_{0}^{\infty}\int_{w}^{\infty} t\varphi(t)dtdw \quad (1)$$

Let $\varphi_0(w) = \varphi(w)$ and $\varphi_n(w) = \int_{w}^{\infty}\varphi_{n-1}(t)dt$ for $n = 1,2,.......$

Note that if $E(W^{n-1})$ exists, then $\varphi_n(w)$ exists and we can write[6]:

$$\varphi_n(w) = \frac{1}{(n-1)!}\sum_{r=0}^{n-1}(-1)^r C_r^{n-1} w^r \int_{w}^{\infty} t^{n-1-r}\varphi(t)dt \quad (2)$$

Moreover since the probability density $\varphi(w)$ vanishes on the negative half of the real line, then $E(W^{n-1}) = (n-1)!\varphi_n(0)$

Using (2) for $n = 2$ we have $\varphi_2(w) = \int_{w}^{\infty} t\varphi(t)dt - w\varphi_1(w)$ (3)

Integrating by parts the last term of the previous expression we get: $\int_{w}^{\infty} w\int_{w}^{\infty}\varphi(t)dtdw = \varphi_3(w)$

Replacing in (3), we get: $2\varphi_3(0) = \int_{0}^{\infty}\int_{w}^{\infty} t\varphi(t)dtdw = E(W^2) = 1$

From (1) we conclude that $\int_{-\infty}^{\infty} h(x)dx = \int_{0}^{\infty}\int_{w}^{\infty} t\varphi(t)dtdw = 1$

We have shown that $h(x)$ is an elliptical density function.

Let Y be the random variable whose density is $h(x)$, since $h(x)$ is symmetric around μ we must have $E(Y) = E(X) = \mu$.

To determine $V(Y)$, let $W = \left|\frac{Y-\mu}{\sigma}\right|$ and notice that $V(Y) = \sigma^2 E(W^2)$.

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[5] See Kelker (1970) P.427.

[5] See Mokhtar (1974) P.543.

$$E\left(W^2\right)=\int_{-\infty}^{\infty}\frac{(y-\mu)^2}{\sigma^2}h(y)dy=\frac{1}{2}\int_{-\infty}^{\infty}w^2\int_{w}^{\infty}t\varphi(t)dtdw=\int_{0}^{\infty}w^2\int_{w}^{\infty}t\varphi(t)dtdw \quad (4)$$

$$\therefore E\left(W^2\right)=\int_{0}^{\infty}w^2\int_{w}^{\infty}t\varphi(t)dtdw=\int_{0}^{\infty}w^2\left(\varphi_2(w)+w\varphi_1(w)\right)dw$$

But $\int_{0}^{\infty}w^2\varphi_2(w)dw=\int_{0}^{\infty}\frac{w^4}{12}\varphi(w)dw=\frac{E(X-\mu)^4}{12\sigma^4}$

Similarly, we have $\int_{0}^{\infty}w^3\varphi_1(w)dw=\int_{0}^{\infty}\frac{w^4}{4}\varphi(w)dw=\frac{E(X-\mu)^4}{4\sigma^4}$

$\therefore E\left(W^2\right)=\frac{E(X-\mu)^4}{3\sigma^4}$, hence $V(Y)=\frac{E(X-\mu)^4}{3V(X)}$ □

Notice that the kurtosis measure of X is given by $K(X)=\frac{V(Y)}{V(X)}$.

For elliptical distributions with heavier tails than the normal we have $K(X)>1$, we must have then $V(Y)>V(X)$.

Furthermore, to evaluate the function $l(x)$ at μ, we evaluate first $h(x)$ at μ to get

$$h(\mu)=\frac{1}{2\sigma}\int_{0}^{\infty}w\varphi(w)dw.$$

Using again the relation $\varphi_2(w)=\int_{w}^{\infty}t\varphi(t)dt-w\varphi_1(w)$

We obtain: $2\sigma h(\mu)=\varphi_2(0)=E\left(\left|\frac{X-\mu}{\sigma}\right|\right)$

We conclude that: $l(\mu)=\frac{h(\mu)}{f(u)}=\frac{E(|X-\mu|)}{2\sigma^2 f(\mu)}$.

A major subclass of elliptical distributions is the variance mixture of normal distributions[7]. It provides attractive and appealing alternatives to normal distributions as a model for the distribution of returns in markets with fat tails.

According to Owen and Rabinovitch (1983), "the density of an elliptically distributed random vector can always be presented as a nondegenerate variance mixture of (at least two) normal densities" (see p.746). Gupta and Varga (1993) give an example (see p.142) which shows that the density of an absolutely continuous univariate elliptical distribution is not always expressible as a scale

[7] For a characterization of variance mixture of normal distributions, see Kelker (1970) and Gupta and Varga (1993).

mixture of normals. Wang (2001) describes a setting under which a linear combination (portfolio) of mixture of normals is a univariate mixture of normals.

We would like to study further the nature of the functions $h(x)$ and $l(x)$ for this particular subclass of elliptical distributions. We show, in particular, that if $f(x)$ is a discrete (continuous) univariate variance mixture of normal distributions then $h(x)$ is a discrete (continuous) variance mixture of normal distributions. Furthermore, since both densities are symmetric around μ, we show that $V(Y) > V(X)$ and $l(\mu) < 1$ hence $h(x)$ is a sort of mean preserving spread of $f(x)$.

If $f(x)$ is a density of a univariate mixture of normal distribution we can write it as follows:

1/ the discrete case: $f\left(x;\mu,\sigma^2\right) = \sum_{i=1}^{m} p_i\varphi_i\left(x;\mu,\sigma_i^2\right)$ Where $\varphi_i\left(x;\mu,\sigma_i^2\right)$ is the density of a normal distribution, $p_i \geq 0\ \forall i$ and $\sum_{i=1}^{m} p_i = 1$. Notice that $\sigma^2 = \sum_{i=1}^{m} p_i\sigma_i^2$

$$\begin{aligned}h(x) &= \int_{-\infty}^{x} \frac{\mu-y}{\sigma^2} f(y)dy = \int_{-\infty}^{x} \frac{\mu-y}{\sigma^2}\left(\sum_{i=1}^{m} p_i\varphi_i\left(x;\mu,\sigma_i^2\right)\right)dy \\ &= \sum_{i=1}^{m} \frac{p_i\sigma_i^2}{\sigma^2}\left(\int_{-\infty}^{x} \frac{\mu-y}{\sigma_i^2}\varphi_i\left(y;\mu,\sigma_i^2\right)dy\right)\end{aligned}$$

Since $\varphi_i'\left(x;\mu,\sigma_i^2\right) = \frac{\mu-x}{\sigma_i^2}\varphi_i\left(x;\mu,\sigma_i^2\right)$ we have $h(x) = \sum_{i=1}^{m} \frac{p_i\sigma_i^2}{\sigma^2}\left(\varphi_i\left(y;\mu,\sigma_i^2\right)\right)$

Therefore the function $h(x)$ is a density of a discrete variance mixture of normals.

2/ the continuous case: $f\left(x;\mu,\sigma^2\right) = \int_0^{\infty} \varphi(x;\mu,z)dG(z)$, where $\varphi(x;\mu,z)$ is the density of a normal distribution with variance Z (see Kelker p426). The function $G(z)$ is the cumulative distribution of Z. Notice that $\sigma^2 = E(Z)$.

$$\begin{aligned}h(x) &= \int_{-\infty}^{x} \frac{\mu-y}{\sigma^2} f(y)dy = \int_{-\infty}^{x} \frac{\mu-y}{\sigma^2}\left(\int_0^{\infty} \varphi(y;\mu,z)dG(z)\right)dy \\ &= \int_0^{\infty} \frac{z}{\sigma^2}\left(\int_{-\infty}^{x} \frac{\mu-y}{z}\varphi(y;\mu,z)dy\right)dG(z)\end{aligned}$$

Since $\varphi'(y;\mu,z)=\frac{\mu-x}{z}\varphi(y;\mu,z)$ we have $h(x)=\int_0^\infty \frac{z}{E(Z)}\varphi(y;\mu,z)\,dG(z)$

Put $\hat{G}(z)=\frac{\int_0^z v\,dG(v)}{\int_0^\infty v\,dG(v)}$ and notice that this is a valid cumulative distribution.

We conclude that $h(x)=\int_0^\infty \varphi(y;\mu,z)\,d\hat{G}(z)$ therefore the function $h(x)$ is a density of a variance mixture of normals.

We can verify by straight forward calculation that $V(X)=E(Z)$ and $E(X-\mu)^4=3E(Z^2)$ similarly $V(Y)=\frac{E(Z^2)}{E(Z)}$, we may conclude that $V(Y)=\frac{E(X-\mu)^4}{3V(X)}$.

When the density of the random variable X takes the form of a variance mixture of normal $f(x;\mu,\sigma^2)=\int_0^\infty \varphi(x;\mu,z)dG(z)$, we have shown that the random variable Y whose density $h(x)$ is of the form of a variance mixture of normals such that $E(X)=E(Y)$.

The function $l(x)=\frac{h(x)}{f(x)}=\frac{\int_0^\infty z\varphi(y;\mu,z)\,dG(z)}{E(Z)\int_0^\infty \varphi(x;\mu,z)dG(z)}$ can be interpreted as a likelihood ratio of two mixtures of normals densities.

Evaluating this function at the point $x=\mu$, we can write $l(\mu)=\frac{E(\sqrt{Z})}{E(Z)E\left(\frac{1}{\sqrt{Z}}\right)}$

To show that $l(\mu)<1$, we note, first, that the function $\psi_1(Z)=\sqrt{Z}$ is concave while the function $\psi_2(Z)=\frac{1}{\sqrt{Z}}$ is convex. On the other hand, by Jensen inequality we have $\sqrt{E(Z)}>E(\sqrt{Z})$ and $\frac{1}{\sqrt{E(Z)}}<E\left(\frac{1}{\sqrt{Z}}\right)$ hence $l(\mu)<1$.

Thus we may say that the function $f(x)$ is more peaked at the center than $h(x)$.
In addition, we note that $V(Y)>V(X)$ since $E(Z^2)>(E(Z))^2$. The random variable X dominates stochastically Y in the sense of Bawa (1975) (see theorem 10 p.116).
We conclude that $h(x)$ is a sort of mean preserving spread of $f(x)$ since both densities are symmetric around μ.
Furthermore, the measure of kurtosis of the random variable X is given by $K(X)=\dfrac{E(X-\mu)^4}{3(V(X))^2}=\dfrac{E(Z^2)}{(E(Z))^2}>1$ which means that the distribution of X has fatter tail than that of a normal.

2. The Portfolio Selection Problem for the Class of Elliptical Distributions

Consider an investor with initial wealth $w_0>0$ to be invested in n risky assets available in the market and indexed by the set $N=\{1,2,........,n\}$, $\forall i \in N$ define:

P_i = the current (certain) price of one unit of asset i

$\widetilde{P}_i$ = the future (uncertain) price of one unit of asset i

α_i = number of units invested in the i^{th} asset

We assume that assets are infinitely divisible and investors are risk-averse expected utility maximizers. Let the set of all at least twice continuously differentiable, increasing and strictly concave functions (risk-averse utilities) be given by $U=\{f \,/\, f'>0\,,\, f''<0\}$.
For $u \in U$, we define the portfolio problem as follows:

$$\max E\left(u\left(\sum_{i\in N}\alpha_i \widetilde{P}_i\right)\right) \quad \text{Subject to } \sum_{i\in N}\alpha_i P_i = w_0$$

The investor is assumed to be maximizing his expected utility of terminal wealth given his initial budget constraint.
Define $X_i=\dfrac{\widetilde{P}_i}{P_i}$ and $a_i=\dfrac{\alpha_i P_i}{w_0}$ $\forall i \in N$ where X_i is one plus the uncertain rate of return on the asset i and a_i is the proportion of wealth invested in asset i.
Define the set of all feasible portfolios: $\Lambda=\{\lambda \in R^n \,/\, \lambda' e=1\}$ where $e \in R^n$ is a vector of ones and prime denotes the transpose of a vector. We allow short sales

here because, among other reasons[8], we want to examine the conditions under which short sales are not an optimal strategy.
In matrix notation, the portfolio problem is: $\max_{a \in \Lambda} E(u(w_0 a'X))$

In what follows, the random vector $X \in R^n$ is assumed to have an elliptical distribution.
The following theorem gives the optimality condition for elliptical distributions. It proved to be useful for establishing many results including some main results stated in Owen and Rabinovitch (1983) and in Chamberlain (1983). In particular, it enables us to characterize the efficient set in the portfolio space analytically and to show that two funds separation holds for elliptical distributions. The theorem provides us also with a global measure of risk aversion that can be used to determine the optimal investment strategy for a risk-averse investor when faced with elliptical risk.

Theorem 1: *Let $X \in R^n$ such that $X \sim \xi(\Delta, \Omega)$. Assume the regularity conditions of the lemma and the covariance matrix $\Sigma = \gamma\Omega$ is non-singular[9]. Then, $\forall u \in U$, the optimal portfolio must satisfy:*

$$a = \frac{\Sigma^{-1}e}{e'\Sigma^{-1}e} + \frac{e'\Sigma^{-1}\Delta}{M(w_0, a'X)}\left(\frac{\Sigma^{-1}\Delta}{e'\Sigma^{-1}\Delta} - \frac{\Sigma^{-1}e}{e'\Sigma^{-1}e}\right)$$ *Where*

$$M(w_0, y) = \frac{-w_0 E(u''(w_0 y)\gamma(w_0 y))}{E(u'(w_0 y))}$$

Proof of theorem 1:
Define the Lagrange function:
$L = E(u(w_0 a'X)) + \lambda(1 - a'e)$, where λ is the Lagrange multiplier.
From the first order conditions of optimality we have:

$$\frac{\partial L}{\partial a_i} = w_0 E(X_i u'(w_0 a'X)) - \lambda = 0 \quad \forall i \in N \quad (1)$$

And $\frac{\partial L}{\partial \lambda} = 1 - a'e = 0$

We have $(n+1)$ equations in $(n+1)$ unknowns
From (1) we can write:

$$w_0\left[\text{cov}(X_i, u'(w_0 a'X)) + E(X_i)E(u'(w_0 a'X))\right] = \lambda \quad \forall i \in N$$

From the lemma we have:

[8] With restriction on short sales, it is quit difficult to characterize the efficient set of portfolios. Dybvig (1985) showed that kinks in the efficient set are the rule rather than the exception if short sales are not allowed.
[9] This means that assets are linearly independent. That is, no asset can be written as a linear combination of other assets.

$$w_0\left(\operatorname{cov}(X_i, w_0 a'X)E(u''(w_0 a'X)l(w_0 a'X)) + E(X_i)E(u'(w_0 a'X))\right) = \lambda \quad \forall i \in N$$

Put $Y = w_0 a'X$, we write in matrix notation:

$$w_0^2 \Sigma a E(u''(Y)l(Y)) + w_0 \Delta E(u'(Y)) = \lambda e$$

Pre multiplying by Σ^{-1}, we get:

$$w_0^2 E(u''(Y)l(Y))a + w_0 \Sigma^{-1} \Delta E(u'(Y)) = \lambda \Sigma^{-1} e \quad (2)$$

Pre multiplying again by e', we get:

$$w_0^2 E(u''(Y)l(Y)) + w_0 e'\Sigma^{-1} \Delta E(u'(Y)) = \lambda e'\Sigma^{-1} e$$

Hence $\lambda = \dfrac{w_0^2 E(u''(Y)l(Y))}{e'\Sigma^{-1}e} + \dfrac{w_0 e'\Sigma^{-1}\Delta E(u'(Y))}{e'\Sigma^{-1}e}$

Rearranging (2) and substituting the value of λ we get:

$$a = \frac{\Sigma^{-1}e}{e'\Sigma^{-1}e} + \frac{e'\Sigma^{-1}\Delta}{M(w_0, a'X)}\left(\frac{\Sigma^{-1}\Delta}{e'\Sigma^{-1}\Delta} - \frac{\Sigma^{-1}e}{e'\Sigma^{-1}e}\right)$$

where $M(w_0, a'X) = \dfrac{-w_0 E(u''(w_0 a'X)l(w_0 a'X))}{E(u'(w_0 a'X))}$ □

The function $M(w_0, y)$ can be interpreted as a global measure of risk aversion. Note that $M(w_0, y) > 0$, to see this it suffices to recall that $h(y) > 0$ and that the utility function is increasing and concave.

It is clear that the relevant measure of global risk aversion in the case of elliptical risk is $M(w_0, y)$. In general, this measure needs to be estimated if exact optimal portfolios are desired. In addition, this new measure generalizes the Rubinstein (1976) measure of risk aversion which is relevant only for the normal subclass of elliptical distribution. To see this set $w_0 = 1$ and note that if Y is normally distributed then $l(y) = 1$ and hence $M(y)$ becomes $R(y) = \dfrac{-E(u''(y))}{E(u'(y))}$ which is the Rubinstein measure.

Furthermore, unlike the Rubinstein measure which depends only on the mean and the variance of the normal distribution, the measure $M(w_0, y)$ depends on the kurtosis of the particular elliptical distribution used as well.

Define the portfolios $a^v = \dfrac{\Sigma^{-1}e}{e'\Sigma^{-1}e}$, $a^t = \dfrac{\Sigma^{-1}\Delta}{e'\Sigma^{-1}\Delta}$ and $\eta = e'\Sigma^{-1}\Delta\left(a^t - a^v\right)$.

The portfolio a^v is the global minimum variance portfolio, a^t is an expected return to risk tradeoff portfolio and η is an arbitrage portfolio that uses no wealth since $\eta' e = 0$.

This theorem implies a main result of Chamberlain (1983), that is, the efficient set (i.e. the set of all optimal risk-averse portfolios) is mean-variance efficient.

Corollary1: *The efficient set of portfolios*[10] *is given by:* $\Lambda^e = \underset{\alpha \in R_+}{\Upsilon}\{a(\alpha)\}$ *where* $a(\alpha) = a^v + \frac{1}{\alpha}\eta$

This set is spanned by the two frontiers portfolios[11] a^v *and* a^t. *Hence the elliptical class implies two funds separation. However, if* $\Delta = ke$ *where* $k \in R$, *this class implies one fund separation.*

In the portfolio space, the efficient set contains the same elements whenever we have the same mean vector Δ and the same covariance matrix Σ regardless of the particular elliptical distribution assumed. Furthermore, under these conditions, the mean variance choices[12] are identical to those of the case of the normal distribution.

If we introduce a riskless asset, in addition to the n risky assets, with one plus rate of return $r_0 < \frac{\Delta'\Sigma^{-1}e}{e'\Sigma^{-1}e}$ then, like in the normal case, the efficient set is spanned by the riskless asset and market portfolio[13] $a^m = \frac{\Sigma^{-1}(\Delta - r_0 e)}{e'\Sigma^{-1}(\Delta - r_0 e)}$ which is the same for all investors given homogenous beliefs about the parameters.
Thus every risk-averse investor with utility function $u \in U$ has an optimal portfolio given by $b^u \in R^{n+1}$ where $b^u = \begin{pmatrix} 1 - a_0^u \\ 0 \end{pmatrix} + a_0^u \begin{pmatrix} 0 \\ a^m \end{pmatrix}$ and a_0^u is the proportion invested in risk. It can be shown[14] that a_0^u is given by: $a_0^u = \frac{1}{M(w_0, x_m)}\left(\frac{\mu_m - r_0}{\sigma_m^2}\right)$, where $x_m = (1 - a_0)r_0 + a_0 r_m$, r_m is the uncertain return of the market portfolio, $\mu_m = E(r_m)$ and $\sigma_m^2 = V(r_m)$

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[10] The set of frontier portfolios is given by: $\Lambda^f = \underset{\alpha \in R}{\Upsilon}\{a(\alpha)\}$ where $a(\alpha) = a^v + \frac{1}{\alpha}\eta$

[11] a^t is a frontier portfolio, it is efficient only if $e'\Sigma^{-1}\Delta > 0$

[12] In the mean standard deviation space, we have the familiar hyperbolic relation between frontier portfolios.

[13] The market portfolio is the unique solution to the mathematical programming problem: $\underset{\alpha > 0}{MAX}\left\{\frac{a'(\alpha)\Delta - r_0}{\sqrt{a'(\alpha)\Sigma a'(\alpha)}}\right\}$ such that $a(\alpha) = a^v + \frac{1}{\alpha}\eta$.

[14] This is a direct result from the first order condition when using the previous lemma.

Notice that a necessary and sufficient condition to have a positive demand for risk is that $\mu_m - r_0 > 0$.

If a risk-averse investor "estimates" correctly the parameters (i.e. Δ and Σ) while he could not "estimate" correctly the functional form of the non normal elliptical distribution (for example the investor might assumes normality), then he will hold an efficient portfolio but with relatively higher (or smaller) risk. On the other hand, if all risk-averse investors agree on fact that the distribution is elliptical, in particular they might disagree about the exact functional form of the distribution and hence the relevant measure of risk, as long as they have homogenous beliefs about the parameters Δ and Ω, their optimal portfolios are elements of Λ^e hence efficient. This fact allowed Owen and Rabinovitch (1983) to derive a generalized CAPM that does not require investors' full agreement upon the specific functional form of the asset returns elliptical distribution. In fact, as long as there is agreement about the parameters of the elliptical distribution(s), all risk-averse investors perceive the same mean-variance efficient set and hence an equilibrium pricing model can be derived[15].

. Unfortunately, showing that the individual optimal portfolio is mean-variance efficient does not imply, in general, that expected utility is independent of higher order moments. The potential sensitivity of expected utility to kurtosis should imply different behavior of risk-averse investors when faced with a non normal elliptical distribution as compared to a normal distribution even though it presents the same mean-variance choices.

Before investigating this line of thought further, we first present the following consequence of the theorem.

Corollary2: *Consider two investors with utility functions $u_1, u_2 \in U$ and assume they have homogenous beliefs about the parameters Δ and Ω , investor i has initial wealth w_i and believes that the functional form of the elliptical distribution is F^i $i = 1,2$.*

Define the two problems:

$$\max_{a \in \Lambda} \int_{x \in R^n} u_i(w_i a'x) dF^i(x) = \max_{a \in \Lambda} E_i(u_i(w_i a'x)) \quad i = 1,2.$$

Let a^* be the optimal solution to P_1, if a^* satisfies the condition:

$$\frac{-w_1 E_1\left(u_1''\left(w_1 X'a^*\right)_1\left(w_1 X'a^*\right)\right)}{E_1\left(u_1'\left(w_1 X'a^*\right)\right)} = \frac{-w_2 E_2\left(u_2''\left(w_2 X'a^*\right)_2\left(w_2 X'a^*\right)\right)}{E_2\left(u_2'\left(w_2 X'a^*\right)\right)}$$

Then a^* is optimal to P_2.

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[15] See Owen and Rabinovitch (1983) for a discussion of this point

For example, let investor 1 have constant absolute risk aversion (CARA) utility function with $\frac{-u''(y)}{u'(y)}=\alpha$ and assume that this investor believes that returns follow a normal distribution with parameters Δ,Σ then the optimal solution[16] for investor 1 is given by $a^*=a^m+\frac{1}{\alpha w_1}\eta$, if in addition $\frac{-w_2E_2\left(u_2''\left(w_2X'a^*\right)_2\left(w_2X'a^*\right)\right)}{E_2\left(u_2'\left(w_2X'a^*\right)\right)}=\alpha w_1$ holds then a^* is optimal for investor 2 as well.

Remark: The result of corollary 2 is similar to a theorem given in Kallberg and Ziemba (9) in the case of normality.

In order to compare the behavior of a risk-averse investor, with initial wealth w_0, when faced with a non normal elliptical distribution instead of a normal distribution that presents the same mean-variance choices, we consider two alternative random vectors of returns $X\sim N(\Delta,\Sigma)$ and $Y\sim\xi(\Delta,\Omega)$. We assume, furthermore, that the covariance of Y is given by $\Sigma=\gamma\Omega$. Given these conditions, the efficient sets under X and under Y are identical regardless of the existence of the riskless asset. The two alternatives present the same mean-variance choices.

If we assume the existence of the riskless asset with sure return $r_0<\frac{\Delta'\Sigma^{-1}e}{e'\Sigma^{-1}e}$ then the market portfolio is the same for both alternatives. It is given, in both cases, by $a^m=\frac{\Sigma^{-1}(\Delta-r_0e)}{e'\Sigma^{-1}(\Delta-r_0e)}$.

Let the end of period wealth be given by $w=w_0\left((1-a_0)r_0+a_0r_m\right)$, where r_m is the uncertain return of the market portfolio, with $\mu_m=E(r_m)$, $\sigma_m^2=V(r_m)$ and a_0 the proportion invested in the risky assets.

We have: $E(w)=w_0\left((1-a_0)r_0+a_0\mu_m\right)$

$$V(w)=w_0^2a_0^2\sigma_m^2$$

$$E(w-E(w))^k=w_0^ka_0^kE(r_m-\mu_m)^k$$

If $r_m\sim N(\mu_m,\sigma_m^2)$ we can show, for $s=\{1,2,3.......\}$, that:

[16] These are the only conditions that imply a closed form solution. See Epps (1981)

$$E(r_m - \mu_m)^k = \begin{cases} \dfrac{(2s)!\left(\sigma_m^2\right)^s}{2^s\, s!} & k = 2s \\ 0 & k = 2s+1 \end{cases}$$

If $r_m \sim \xi\left(\mu_m, \sigma_m^2\right)$ with density given by $f\left(x;\mu,\sigma_m^2\right) = \int_0^\infty \varphi(x;\mu,z)\,dG(z)$, where $\varphi(x;\mu,z)$ is the density of a normal distribution with variance Z and $E(Z) = \sigma_m^2$, then, for $s = \{1,2,3......\}$, we have:

$$E(r_m - \mu_m)^k = \begin{cases} \dfrac{(2s)!\,E\left(Z^s\right)}{2^s\, s!} & k = 2s \\ 0 & k = 2s+1 \end{cases}$$

Let the end of period wealth be x_m when $r_m \sim N\left(\mu_m, \sigma_m^2\right)$ and y_m when $r_m \sim \xi\left(\mu_m, \sigma_m^2\right)$, then we have:

$$\frac{E(y_m - E(y_m))^k}{E(x_m - E(x_m))^k} = \frac{E\left(Z^s\right)}{(E(Z))^s} > 1 \qquad \forall s \geq 2 \text{ and } k = 2s$$

The expression above is greater than one since the function $\psi(Z) = Z^s$ is convex for $Z > 0$ and $s \geq 2$, hence by Jensen inequality we have $E\left(Z^s\right) > (E(Z))^s$.

Define the two problems: P_1: $\max\limits_{a_0} \{\varphi_1(a_0) = E(u(x_m))\}$

P_2: $\max\limits_{a_0} \{\varphi_2(a_0) = E(u(y_m))\}$

The function $u \in U$ is assumed to exhibit consistent risk aversion. A utility function exhibits consistent risk aversion if the k^{th} derivative $u^{(k)}(w)$ has uniformly the same sign $\forall w$. For risk-averse utility functions, Scott and Horvath (1980) proved that $u^{(k)}(w) > 0$ for odd k and $u^{(k)}(w) < 0$ for even k. In particular, the contribution of even central moments to expected utility in a Taylor expansion around the mean is negative. Examples of utilities exhibiting consistent risk aversion include the logarithmic, the power and the CARA utilities.

We write the expected utilities in the problems P_1 and P_2 as a Taylor expansion around the mean as follows:

$$E(u(x_m)) = u(E(x_m)) + \sum_{k=1}^{\infty} \frac{w_0^{2k}\, a_0^{2k}\, (2k)!}{2^k\, k!} (E(Z))^k\, u^{(2k)}(E(x_m)) \quad (1)$$

$$E(u(y_m)) = u(E(y_m)) + \sum_{k=1}^{\infty} \frac{w_0^{2k} a_0^{2k} (2k)!}{2^k k!} E(Z^k) u^{(2k)}(E(y_m)) \quad (2)$$

Notice that $E(x_m) = E(y_m) = w_0((1-a_0)r_0 + a_0 \mu_m) = E(w)$

Furthermore, since $E(Z^k) > (E(Z))^k$ and $u^{(k)}(w) < 0$ for even k, we have:

$$E(u(x_m)) - E(u(y_m)) = \sum_{k=1}^{\infty} \frac{w_0^{2k} a_0^{2k} (2k)!}{2^k k!} \{(E(Z))^k - E(Z^k)\} u^{(2k)}(E(w)) > 0$$

This means that x_m dominates stochastically y_m for the class of utility functions exhibiting consistent risk aversion.

Let a_0^* be the optimal solution to P_1 and define $w^* = w_0((1-a_0^*)r_0 + a_0^* r_m)$, we must have:

$$\frac{\partial}{\partial a_0}(E(u(x_m))) = 0$$

$$w_0(\mu_m - r_0)u'(E(w^*)) + \sum_{k=1}^{\infty} \frac{w_0^{2k}(2k)!(E(Z))^k}{2^k k!} \frac{\partial}{\partial a_0}\left((a_0^*)^{2k} u^{(2k)}(E(w^*))\right) = 0 \quad (3)$$

Evaluating the first derivative of (2) at a_0^* we have:

$$\frac{\partial}{\partial a_0}(E(u(y_m))) =$$

$$w_0(\mu_m - r_0)u'(E(w^*)) + \sum_{k=1}^{\infty} \frac{w_0^{2k}(2k)!(E(Z^k))}{2^k k!} \frac{\partial}{\partial a_0}\left((a_0^*)^{2k} u^{(2k)}(E(w^*))\right) < 0 \quad (4)$$

Expression (4) is negative because of (3) and the fact that $E(Z^k) > (E(Z))^k$.

Let a_0^{**} be the optimal solution to P_2, then from (4) we conclude that $a_0^{**} < a_0^*$.

The forgone opportunity cost of mean-variance investment strategies was investigated by Levy and Morkowitz (1979) and Simaan (1993b) among others. Levy and Morkowitz (1979) found that CARA utilities provided the worst mean-variance approximation to expected utility among all the utility functions employed. Simaan (1993b) used CARA utilities and a parametric joint distribution where idiosyncratic security risks are modeled as following a joint normal distribution with a single factor having different loadings to model the skewed noise. When a riskless asset is introduced, he found that the optimization premium is at least ten times lower than the typical management fee regardless of the degree of relative risk aversion. The forgone opportunity cost of mean-variance in Simaan (1993b) is due to the ignored skewness parameter, what is needed in our setting, however, is an investigation of the forgone opportunity cost within the mean-variance framework due to the ignored kurtosis parameter. While the optimization premium due to ignoring skewness was found in Simaan (1993b) to be negligible compared to transaction costs, there is enough empirical evidence[17] to believe that this is not the case when ignoring kurtosis.

Let the investor has a CARA utility function with $\frac{-u''(y)}{u'(y)} = \alpha$.

If $r_m \sim N(\mu_m, \sigma_m^2)$, then under x_m the optimal portfolio is given by $a_0^* = \frac{\mu_m - r_0}{\alpha w_0 \sigma_m^2}$.

If we assume instead that r_m follows a variance mixture of two normals with density given by $f(x;\mu_m,\sigma_m^2) = p\varphi(x;\mu_m,\sigma_1^2) + (1-p)\varphi(x;\mu_m,\sigma_2^2)$, where $\varphi(x;\mu,\sigma^2)$ is the density of a normal distribution with variance σ^2, then y_m is a variance mixture of two normals with $E(y_m) = w_0((1-a_0)r_0 + a_0\mu_m)$ and $V(y_m) = w_0^2 a_0^2 \sigma_m^2$.

Under y_m however, the investor's optimal portfolio must satisfy: $a_0 = \frac{1}{M(w_0, y_m)}\left(\frac{\mu_m - r_0}{\sigma_m^2}\right)$ where

$$M(w_0, y_m) = \alpha w_0 \frac{\theta \exp(0.5\alpha^2 w_0^2 a_0^2 \sigma_1^2) + (1-\theta)\exp(0.5\alpha^2 w_0^2 a_0^2 \sigma_2^2)}{p \exp(0.5\alpha^2 w_0^2 a_0^2 \sigma_1^2) + (1-p)\exp(0.5\alpha^2 w_0^2 a_0^2 \sigma_2^2)},$$

and $\theta = \frac{p\sigma_1^2}{p\sigma_1^2 + (1-p)\sigma_2^2}$.

Notice that when $\sigma_1^2 > \sigma_2^2$, we have $\theta > p$. Similarly, when $\sigma_1^2 < \sigma_2^2$, we have $(1-\theta) > (1-p)$. We conclude, in both cases, that $M(w_0, y_m) > \alpha w_0$ for all $a_0 > 0$.

Hence the optimal solution a_0^{**} under y_m satisfies $a_0^{**} < a_0^*$.

To shed more light on the role of kurtosis in determining the optimal investment strategy, we consider an example[18] provided by Mukhtar (1974).

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[17] Duffie and Pan (1997) states that, for many markets, returns have fatter than normal tails. In particular, the S&P500 daily returns from 1986 to 1996 have an extremely high sample kurtosis of 111 and negative skewness of -4.81. Longin (1995) studied the behavior of daily returns in the French stock market over the period 1977-1990 and found deviation from normality with high kurtosis of 6.294 and relatively small negative skewness of -0.532. The Indian stock market was studied by Broca (2002) for the period 1985-1998.The empirical series displays non normality with severe kurtosis of 5.046 and slight positive skewness of 0.149.

[18] See p.543 in Mukhtar (1974).

Let X_k, for $k \in \{2,3,.....\}$, a sequence of random variables with densities given by:
$f(x;\mu,2)=\left(1-\frac{1}{k^2-1}\right)\varphi(x;\mu,1)+\left(\frac{1}{k^2-1}\right)\varphi(x;\mu,k^2)$, where $\varphi(x;\mu,\sigma^2)$ is the density of a normal distribution with mean μ and variance σ^2

Notice that $E(X_k)=\mu$ and $V(X_k)=2 \;\; \forall k$. As pointed out by Mukhtar (1974), X_k follows a scale mixture of normal distributions which converges weakly to $\varphi(x;\mu,1)$.

However for sufficiently large k, the kurtosis measure $K(X_k)=\frac{E(X_k-\mu)^4}{3(V(X_k))^2}=\frac{1}{4}(2+k^2)$ is quite large compared to that of the normal distribution.

If we reconsider the previous investor's problem with the uncertain return of the market portfolio being given by X_k, we have then:

$$M_k(w_0, y_m)=\alpha w_0\frac{(k^2-2)+k^2\exp(0.5\alpha^2 w_0^2 a_0^2(k^2-1))}{2(k^2-2)+2\exp(0.5\alpha^2 w_0^2 a_0^2(k^2-1))}$$

Let $a(k)$ be the investment in risk, we have:

$$a(k)=\frac{(k^2-2)\exp(-0.5\alpha^2 w_0^2(k^2-1)(a(k))^2)+1}{(k^2-2)\exp(-0.5\alpha^2 w_0^2(k^2-1)(a(k))^2)+k^2}\left(\frac{\mu-r_0}{\alpha w_0}\right) \quad (1)$$

Notice that $\forall k\in\{2,3,.....\}$, $0<a(k)<\frac{\mu-r_0}{2\alpha w_0}$. Hence, with the uncertain return of the market portfolio being given by X_k, $\forall k\in\{2,3,.....\}$, the investment in risk $a(k)$ is smaller compared to both the limiting normal distribution $\varphi(x;\mu,1)$ with optimal solution $a_0=\frac{\mu-r_0}{\alpha w_0}$ and to the normal distribution $\varphi(x;\mu,2)$ which has the same mean and the same variance as X_k with optimal solution $a_0=\frac{\mu-r_0}{2\alpha w_0}$.

Furthermore, for sufficiently large k, $a(k)$ is sufficiently small. In particular, we see that the limit $\lim_{k\to\infty} a(k)$ must be 0. For suppose that $\lim_{k\to\infty} a(k)=l>0$, replacing in (1) and taking limits we get $l=0$ which is a contradiction.

The sensitivity of expected utility to kurtosis and the other higher even central moments imply that a risk-averse investor demand for risk is smaller when faced with a fatter tails elliptical distribution instead of a normal distribution that presents the same mean-variance choices.

If there is no riskless asset, using similar arguments as above, we can show that the optimal portfolio of a risk-averse investor is closer to the minimum variance portfolio when faced with a non normal elliptical distribution with fat tails instead of a normal distribution that presents the same mean-variance choices.

Many authors (See for example Brumelle, 1974 and Scheffman, 1975) were concerned about conditions on the distributions of asset returns under which diversification is the optimal investment strategy for all risk adverse investors. Mostly in a two assets world, they gave sufficient conditions for positive diversification i.e. holding assets long. These conditions have been associated with the notion of negative interdependence between assets.

In general, negative correlation is neither necessary nor sufficient for diversification (See Brumelle, 1974). However, when returns follow an elliptical distribution the usual correlation coefficient is all that is needed as model of dependence, to study diversification. The next theorem generalizes a known result for normal distributions[19] to the class of elliptical distributions. It gives necessary and sufficient conditions for short sales not to be an optimal investment strategy for all risk-averse investors.

Theorem 2: *Let $X \in R^n$ such that $X \sim \xi(\Delta, \Omega)$ where Ω is non singular, then all risk adverse investors hold non negative (positive) amounts of each asset if and only if:*

I) $\Delta = ke$ *for some* $k \in R$

ii) $\Omega^{-1}e \geq (>) 0$

Proof of theorem 2:

Without loss of generality set $w_0 = 1$. According to theorem 1, $\forall u \in U$ the optimal solution satisfies: $a^u = a^v + \dfrac{1}{M_u(a'X)}\eta$

Where $a^v = \dfrac{\Omega^{-1}e}{e'\Omega^{-1}e}$, $a^t = \dfrac{\Omega^{-1}\Delta}{e'\Omega^{-1}\Delta}$ and $\eta = e'\Omega^{-1}\Delta\left(a^t - a^v\right)$, and $M_u(y) = \dfrac{-E(u''(y)l(y))}{E(u'(y))}$.

In particular, for constant absolute risk aversion (CARA) utility functions with $\dfrac{-u''(y)}{u'(y)} = \alpha$, $\alpha \in R_+$, we have $M_u(y) = M_\alpha(y) = \dfrac{-\alpha E\left(e^{-\alpha y} l(y)\right)}{E\left(e^{-\alpha y}\right)}$.

1/ If $\Delta = ke$ then $a^v = a^t$ and $\eta = 0$. Thus we have $a^u = a^v = \dfrac{\Omega^{-1}e}{e'\Omega^{-1}e}$ $\forall u \in U$.

[19] I am unable to provide a reference even though I should think it must have been stated somewhere.

But $e'\Omega^{-1}e \geq 0$ since Ω^{-1} is positive definite, hence $\Omega^{-1}e \geq 0$ implies $a^u \geq 0 \;\; \forall u \in U$

2/ If $a^u \geq 0 \;\; \forall u \in U$, then $a^\alpha = a^v + \dfrac{1}{M_\alpha(a'X)}\eta \geq 0 \;\; \forall \alpha \in R_+$, in scalar notation

we have $a_j^\alpha = a_j^v + \dfrac{1}{M_\alpha(a'X)}\eta_j \geq 0 \;\; \forall \alpha \in R_+$

We show in this case that $\eta = 0$. Suppose $\eta \neq 0$, then there exists $j \in N$ such that $\eta_j < 0$, note that $\lim_{\alpha \to 0} M_\alpha(y) = 0$ implies $a_j < 0$ contrary to our assumption, hence $\eta = 0$.

But $\eta = 0$ implies $\dfrac{\Omega^{-1}\Delta}{e'\Omega^{-1}\Delta} = \dfrac{\Omega^{-1}e}{e'\Omega^{-1}e}$ hence $\Delta = \dfrac{e'\Omega^{-1}\Delta}{e'\Omega^{-1}e}e = ke$

Furthermore, if $\Delta = ke$ then $a^u = \dfrac{\Omega^{-1}e}{e'\Omega^{-1}e} \;\; \forall u \in U$ but this is nonnegative by assumption.

Since $e'\Omega^{-1}e \geq 0$ we must have $\Omega^{-1}e \geq 0$. □

For elliptical distributions, according to the theorem, short sales are not the optimal investment strategy for all risk-averse investors provided that means of asset returns are equal and the inverse of the variance-covariance matrix has non negative (positive) row sums.

Conclusion

For many markets, previous empirical studies have shown that returns have fatter than normal tails with an extremely high sample kurtosis. The class of elliptical distributions provides attractive and appealing alternatives to normal distributions in modeling the empirical distribution of returns since their densities are more flexible than the normal density because their tails could be longer or shorter. We derived a lemma related to the covariance of functions of elliptical random variables. Based on this result, we obtained the conditions of optimality for the portfolio choice problem and showed how to derive many useful results for the elliptical class of distributions including some main results in the studies of Chamberlain (1983) and of Owen and Rabinovitch (1983). In particular, the efficient set, in the portfolio space, is obtained analytically. Furthermore, a general global risk aversion measure which generalizes the Rubinstein's measure was derived and shown to be relevant to the case of elliptical risk. This new measure was shown to be sensitive to the fatness of tails of the distribution of asset returns.
If a riskless asset exists, the sensitivity of expected utility to kurtosis (and the other higher even central moments) implies that a risk-averse investor demand for risk is smaller when faced with a fatter tails elliptical distribution instead of a normal

distribution that presents the same mean-variance choices. When the kurtosis measure gets sufficiently large, a risk-averse investor tends to invest all his wealth in the riskless asset even when the variance remains constant. While the forgone opportunity cost of mean-variance investment strategies due to ignoring skewness was found in Simaan (1993b) to be negligible compared to typical transaction costs, there is enough empirical evidence to believe that the forgone opportunity cost within the mean-variance framework due to the ignored kurtosis is not negligible.

Finally, a general diversification result for elliptical distributions was derived. We showed that short sales are not an optimal investment strategy for all risk-averse investors if and only if the means of asset returns are equal and the inverse of the variance-covariance matrix has non negative (positive) row sums.

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