Using Bayesian approach to study robustness of classes of priors for homogeneous and non homogeneous Poison processes

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Abstract

A Non-Homogeneous process is a process with rate parameter λ(t) such that this rate is a function of time. Bayesians are interested in robustness with respect to changes in prior distributions/sampling models/loss functions. In This work, we focused on replacing a single prior distribution by a class of priors of the parameters of a given Poisson processes, and developing methods of computing the range of the ensuing answers as the prior varied over the class. This approach, called "global robustness".

Key words: Bayesian Robustness, Nonhomogeneous Poisson processes, Prior Robustness, Global Sensitivity, Local Sensitivity.

Résumé

Un processus de Poisson non homogène est un processus avec le paramètre λ(t) qui est une fonction de temps. Les Bayésiens sont intéresses par la robustesse en ce qui concerne les changements des distributions a priori/des modèles/des fonctions de perte. Dans ce travail, on s'intéresse à remplacer une loi a priori par une classe de lois a priori des paramètres d'un processus de Poisson donné, et de développer le calcul du rang quand la loi a priori change dans cette classe, cette approche s'appelle " la robustesse globale ".

Mots clés : Robustesse Bayésienne, Processus de Poisson Non homogène, Robustesse des lois a priori, sensitivité globale, sensitivité locale.

I- Introduction

There are two main philosophical approaches to statistics. The first is often referred to as the frequentist approach. Sometimes it is called the classical approach. The frequentist approach to statistics considers the parameter to be a fixed but unknown constant. The alternative approach is the Bayesian approach, the Bayesian approach to statistics allows the parameter to be considered a random variable. Probabilities can be calculated for parameters as well as observations and sample statistics. Probabilities calculated for parameters are interpreted as degree of belief and must be subjective.

Robust Bayesian Analysis is concerned with the sensitivity of the results of a Bayesian analysis to the inputs for the analysis. Intuitively, robustness means lack of sensitivity of the decision or inference to assumptions in the analysis that may involve a certain degree of uncertainty. In an inference problem, the assumptions usually involve choice of the model and prior, whereas in a decision problem there is the additional assumption involving the choice of the loss or utility function. An analysis to measure the sensitivity is called sensitivity analysis. Clearly, robustness with respect to all three of these components is desirable. That is to say that reasonable variations from the choice used in the analysis for the model, prior, and loss function do not lead to unreasonable variations in the conclusions arrived at.

In this work, the model proposed is the nonhomogeneous Poisson processes(NHPPs), which have relevant applications in many fields, e.g. in reliability, capturing different behaviors of the problem they are modeling and providing different results (estimations and forecasts). A thorough study of robustness associated to NHPPs is still missing and the proposed work aims to pursue such goal.

The objective of this work is then to study the robustness of the classes of priors of the parameters of a given NHPP where the priors changes over this proposed class and computation of some measures of robustness. Computations with NHPP are difficult, we start with homogeneous Poisson processes and then we will do it later with NHPP.

II- Methods and discussion

The Poisson distribution is used to count the number of occurrences of rare events which are occurring randomly through time (or space) at a constant rate. The events must occur one at a time. We study the robustness of the classes of priors of the parameters of a given NHPP where the prior changes over a proposed class and computation of some measures of robustness. Computations with NHPP are difficult. we start with homogeneous Poisson processes and then we will do it later with NHPP. A detailed discussion on the results of Bayesian robustness analysis of the classes proposed of homogeneous and nonhomogeneous Poisson processes is provided.

1. Study of Robustness - class of prior for HPP

1.1.Class of Gamma(α, β)

Let π (λ) a Gamma prior corresponding ti the likelihood$f(t_1, t_2, \dots, t_n)$, and define

$$\Gamma = \left\{\pi:\ \ Gamma(\alpha,\beta): \frac{\alpha}{\beta} = k\right\}$$

Class of all Gamma priors with mean k, where k is fixed, which can be written also as

$$\Gamma = \{\pi:\ \ Gamma(k\beta,\beta): \beta > 0\}$$

By this class Γ of prior, the posterior distribution is a $Gamma\ (k\beta +\ n, \beta +\ T)$

The quantity of interest is the posterior mean,

$$E(\lambda/t) = h(\beta) = \frac{k\beta + n}{\beta + T}$$

The uncertainty might be quantified by specifying the range spanned by the posterior mean, as the prior varies over the class. Let:

$$\acute{h}(\beta) = \frac{kT - n}{(\beta + T)^2}$$

We are

$$\acute{h}(\beta) > 0 \; if \; kT - n > 0 \; \Rightarrow \sup_{\beta > 0} E(\lambda/t) \underset{\beta\to\infty}{=} k \; and \; \inf_{\beta > 0} E(\lambda/t) \underset{\beta\to\infty}{=} \frac{n}{T}$$

$$\acute{h}(\beta) < 0 \; if \; kT - n < 0 \; \Rightarrow \sup_{\beta > 0} E(\lambda/t) \underset{\beta\to\infty}{=} \frac{n}{T} \; and \; \inf_{\beta > 0} E(\lambda/t) \underset{\beta\to\infty}{=} k$$

$$\acute{h}(\beta) = 0 \; if \; kT - n = 0 \; \Rightarrow \; E(\lambda/t) = k \; is \; constant$$

Then we use the global measures of sensitivity for determining the range of the posterior mean as the prior varies over the class

$$rang = \sup_{\pi\in\Gamma} E(\lambda/t) - \inf_{\pi\in\Gamma} E(\lambda/t) = \left|k - \frac{n}{T}\right|$$

If the measure of range is small then robustness is achieved and any prior in the class can be chosen; and if the measure of range is large there is not robustness. We consider a numerical example; data were generated by a Poisson Process assuming$\lambda = 2$. The obtained data are shown in table 01.

Table 01 : Data generated by a Poisson Process assuming λ= 2.

n	1	2	3	4	5	6	7	8	9	10
t_i	0.1754	0.7840	1.7661	2.4653	3.0960	4.3574	4.3660	5.1674	5.1941	5.6102
n	11	12	13	14	15	16	17	18	19	20
t_i	6.5730	7.0425	7.2365	7.8401	8.5844	9.0439	9.2199	9.9472	10.2435	10.7079

Resulting range for different values of K are presented in table 02. For values of k inferior than 1.7 and superior than 1.9 the range is large and the robustness is not achieved. The robust situation is when k = 1.82. Then, any prior $G(\alpha,\beta)$ in the class Γ can be chosen and there is robustness.

Table 02: range for posterior mean for Gamma prior.

K	max	Min	Range	|range|	K	max	min	range	|range|
0.10	1.8182	0.10	-1.7182	1.7182	1.81	1.8182	1.81	-0.0082	0.0082
0.15	1.8182	0.15	-1.6682	1.6682	1.82	1.82	1.8182	0.0018	0.0018
0.24	1.8182	0.24	-1.5782	1.5782	1.84	1.84	1.8182	0.0218	0.0218
0.35	1.8182	0.35	-1.4682	1.4682	1.88	1.8	1.8182	0.0618	0.0618
0.45	1.8182	0.45	-1.3682	1.3682	1.90	1.90	1.8182	0.0818	0.0818
0.76	1.8182	0.76	-1.0582	1.0582	2.05	2.05	1.8182	0.2318	0.2318
0.85	1.8182	0.85	-0.9682	0.9682	2.36	2.36	1.8182	0.5418	0.5418
1.15	1.8182	1.15	-0.6682	0.6682	2.55	2.55	1.8182	0.7318	0.7318
1.43	1.8182	1.43	-0.3882	0.3882	2.85	2.85	1.8182	1.0318	1.0318
1.70	1.8182	1.70	-0.1182	0.1182	3.20	3.20	1.8182	1.3818	1.3818

1.80	1.8182	1.80	-0.0182	0.0182	8.10	8.10	1.8182	6.2818	6.2818

1.2 Class based on median at $\lambda = 2$ and quartiles at 1 and 3:

We observe N(t), a homogeneous Poisson process with parameter λ. Further, it is felt a priori that λ has a distribution with median 1, i.e.

$$P^{\pi}(\lambda \leq 2) = 0.5 = P^{\pi}(\lambda \geq 2)$$

Upper quartile 3 and under quartiles 1:

$$P^{\pi}(\lambda \leq 1) = P^{\pi}(1 \leq \lambda \leq 2) = P^{\pi}(2 \leq \lambda \leq 3) = P^{\pi}(\lambda \geq 3) = 0.25$$

The class is defined as

$$\Gamma_Q = \{\pi : quartiles\ at\ 1\ and\ 3, median\ at\ 2\}$$

The quantity of interest is always the posterior mean. Then, under Γ_Q, the posterior mean is defined

$$E(\lambda / t) = \frac{\int_0^{\infty} \lambda . f(t,\lambda).\pi(\lambda)\, d\lambda}{\int_0^{\infty} f(t,\lambda).\pi(\lambda)\, d\lambda} = \frac{\int_0^{\infty} \lambda^{n+1} e^{-\lambda T}.\pi(\lambda)\, d\lambda}{\int_0^{\infty} \lambda^{n} e^{-\lambda T}.\pi(\lambda)\, d\lambda}$$

A natural global measure of sensitivity of the Bayesian quantity to the choice of prior is the range of this quantity of interest as the prior varies in the class of priors.

$$\sup_{\pi \in \Gamma_Q} E(\lambda/t) - \inf_{\pi \in \Gamma_Q} E(\lambda/t)$$

We consider a discrete distribution in four points $\lambda_1; \lambda_2; \lambda_3; and\ \lambda_4$ (a DIRAC distribution), we obtain :

$$\underset{\lambda_1\in[0,1],\lambda_2\in[1,2],\lambda_3\in[2,3],\lambda_4\in[3,\infty]}{sup} \frac{\sum_{i=1}^{4} \lambda_i^{\,n+1} e^{\lambda_i T}}{\sum_{i=1}^{4} \lambda_i^{\,n} e^{\lambda_i T}} - \underset{\lambda_1\in[0,1],\lambda_2\in[1,2],\lambda_3\in[2,3],\lambda_4\in[3,\infty]}{Inf} \frac{\sum_{i=1}^{4} \lambda_i^{\,n+1} e^{\lambda_i T}}{\sum_{i=1}^{4} \lambda_i^{\,n} e^{\lambda_i T}}$$

We must find the values of $\lambda_1;\ \lambda_2;\ \lambda_3; and\ \lambda_4$ who minimize and maximize the quantity of the posterior mean under this class.

Considering the same data generated by Poisson process, and the class of priorΓ_Q. We get the upper and the lower bounds of the posteriors means in this class, the results of the optimizations are presented in Table 03.

Table 03: Range of posterior mean for the class of priors Γ_Q.

λ_1	λ_2	λ_3	λ_4	max	λ_1	λ_2	λ_3	λ_4	min	$rang$
0.2	1.4355	2	5.6364	38.2112	0.2	1.9743	3	5.4702	1.7766	36.4346
0.1999	1.9220	3	5.5759	39.5874	0.2	1.2866	3	6.4040	1.5066	38.0808
0.2517	1.9679	3	8	38.4052	1	1.4231	2	9.2154	1.7505	36.6540
0.2455	1.9163	3	28	39.7090	1	1.2454	3	29.3548	1.4431	38.2659

After a choice of several values initials of $\lambda_1;\ \lambda_2;\ \lambda_3; and\ \lambda_4$, the difference between upper and lower bounds on the quantity of interest is too large. (Its value measures the variation caused by the uncertainty in the prior).

2. Study of Robustness - class of prior for NHPP

2.1. Nonhomogeneous Poisson Process

Nonhomogeneous Poisson process (NHPP) is a Poisson process whose intensity function is not a constant. A counting process N(t), t≥0 is a nonhomogeneous Poisson process if:

- N(0)=0
- The process has independent increments and stationary increments;
- $P[N(t+dt)-N(t)=k]=0+o(dt),\quad k\geq 2$
- $P[N(t+dt)-N(t)=k]=\lambda(t)dt+o(dt),\ k=1$
- $P[N(t+dt)-N(t)=k]=1-\lambda(t)dt+o(dt),\quad k=0.$

A homogeneous Poisson process may be viewed as a special case when $\lambda(t) = \lambda$, a constant rate. The mean value function (m.v.f.) of the NHPP is defined as the nondecreasing, nonnegative function:

$$M(s,t) = E\{N(s,t)\}, 0 \le s < t \, with \, M(t) = E\{N(t)\}, t \ge 0$$

Where

$$M(s,t) = \int_{y}^{s} \lambda(t)dt$$

Nonhomogeneous Poisson processes (NHPP) are widely used as models for failures of a repairable system.

2.2. Class based on the Cox-Lewis process

The Cox-Lewis process is one of the general classes of a NHPP, this class is defined by the intensity function $\lambda(t,\alpha,\beta) = \alpha exp\{\beta t\}$. The class proposed the following:

$$\Gamma_C = \{all \; NHPP \; with \quad \lambda(t) = \lambda e^{+\theta t}, \;\; \theta \in [-1,+1], \;\; \lambda \sim G(\alpha,\beta)\}$$

Where, $\lambda(t) = \lambda e^{+\theta t}$ is the intensity function of the Cox-Lewis process.

Consider a NHPP with intensity function λ(t). Suppose we observe the system up to time T and let n be the number of occurrences, occurred at times $t_1, t_2, \dots, t_n$; then the likelihood function is given by:

$$L(t_1, t_2, \dots, t_n) = \prod_{i=1}^{n} \lambda(t_i) exp\left\{-\int_{0}^{T} \lambda(t)dt\right\}$$

$$L(t_1, t_2, \dots, t_n) = \lambda^n exp\left\{\theta \sum_{i=1}^{n} (t_i)\right\} exp\left\{-\frac{\lambda}{\theta}(e^{\theta T} - 1)\right\}$$

A conjugate prior distribution for $\lambda$is given by $\lambda \backsim G(\alpha,\beta)$, then

$$\pi(\lambda)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\lambda\beta}$$

By applying the Bayes theorem we obtain

$$\pi(\lambda/t)\propto\lambda^{\alpha+n-1}.exp\left\{-\lambda(\beta+\frac{e^{\theta T}-1}{\theta})\right\}$$

$$(\lambda/t)\backsim G(\alpha+n,\beta+\frac{e^{\theta T}-1}{\theta})$$

The Posterior mean is:

$$E(\lambda/t)=\frac{\alpha+n}{\beta+\frac{e^{\theta T}-1}{\theta}}$$

The global robustness is based on the calculation of the range of the quantity of interest (posterior mean) when the prior change over this class

$$\sup_{\pi\in\Gamma_C}E(\lambda/t)-\inf_{\pi\in\Gamma_C}E(\lambda/t)$$

$$\sup_{\pi\in\Gamma_C}E(\lambda/t)=\sup_{\theta\in[-1,+1]}E(\lambda/t)=\frac{\alpha+n}{\beta+\inf\limits_{\theta\in[-1,+1]}\frac{e^{\theta T}-1}{\theta}}$$

$$\inf_{\pi\in\Gamma_C}E(\lambda/t)=\inf_{\theta\in[-1,+1]}E(\lambda/t)=\frac{\alpha+n}{\beta+\sup\limits_{\theta\in[-1,+1]}\frac{e^{\theta T}-1}{\theta}}$$

We put $h(\theta)=\frac{e^{\theta T}-1}{\theta}$, we must find $\hat{\theta}_1$ which minimise $h(\theta)$ and maximize $E(\lambda/t)$, then we must find $\hat{\theta}_1$ which maximize $h(\theta)$ and minimize $E(\lambda/t)$.

Considering the same data generated by Poisson process, we get the upper and the lower bounds of the posteriors means in this class, the results of the optimizations for different values of α and β are presented in table 04.

Table04: Range of posterior mean for the class of priors Γ_C.

θ_{max}	$\sup_{\pi\in\Gamma_C} E(\lambda/t)$	θ_{max}	$\inf_{\pi\in\Gamma_C} E(\lambda/t)$	Range
-0.9999	11.2219	1	0.000337	11.2216
-0.9999	10.0998	1	0.000337	10.0994
-0.9999	6.7332	1	0.000337	6.7329
-0.9999	6.6699	1	0.000334	6.6696
-0.9999	4.0002	1	0.000334	3.9908
-0.9999	2.8571	1	0.000334	2.8568
-0.9999	1.8182	1	0.000334	1.8178
-0.9999	0.9524	1	0.000334	0.9520
-0.9999	0.3922	1	0.000334	0.3918
-0.9999	0.3279	1	0.000334	0.3275

For small values of α and a large values of β of Gamma distribution, the range of posterior mean is reasonably small. The conclusion of this analysis would be that robustness likely obtains for smaller values of α and larger values of β .

III- Conclusion

The Poisson process has found numerous applications in science, engineering, economics and other areas. The NHPP is probably the best known generalization of the Poisson process. It is characterized by a deterministic intensity function that describes how the rate of the process changes in time. For an ordinary Poisson processes, this function is a constant.

Robust Bayesian Analysis is concerned with the sensitivity of the results of a Bayesian analysis to the inputs for the analysis. Intuitively, robustness means lack of sensitivity of the decision or inference to assumptions in the analysis that may involve a certain degree of uncertainty. In an inference problem, the assumptions usually involve choice of the model and prior, whereas in a decision problem there is the additional assumption involving the choice of the loss or utility function. An analysis to measure the sensitivity is called sensitivity analysis. Clearly, robustness with respect to all three of these components is desirable. That is to say that reasonable variations from the choice used in the analysis for the model, prior, and loss function do not lead to unreasonable variations in the conclusions arrived at.

In most cases in practice, quantification of subjective belief or judgment is not easily available. It is then common to choose from among conventional priors on the basis of some relatively simple subjective judgments about the problem and the conventional probability model for the data. Such priors have been criticized for various reasons.

We proposed in this work, a Poisson process model and studied the global robustness of some class of priors for the parameters of NHPP and HPP. In perspective, it will be interest to study the local sensitivity of some classes of priors for the parameters of a given NHPP.

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