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Investigating the Efficacy of Various Parameter Estimation Methods for the Gamma Distribution

Brahim Taoussi[1], *National School of Commerce (ESC)*

B_taoussi@esc-alger.dz

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Abstract

The gamma distribution is highly regarded in probability and statistics, widely applied for modeling diverse phenomena such as waiting times, income distribution, insurance severity claims, and wind speed. In this study we explored the accuracy of four statistical parameter estimation methods, namely the maximum likelihood method (MLM), method of moments (MOM), least squared method (LSM), and weighted least squared method (WLSM), in estimating the shape and scale parameters of the gamma distribution through Monte-Carlo simulation across various sample sizes and different combinations of shape and scale parameters values. The Monte-Carlo simulation results indicated that MLM tends to overestimate the shape parameter, while LSM and WLSM are the most unbiased for the shape parameter. Regarding the scale parameter, MOM outperforms LSM in estimating it with a median bias close to zero and moderately low median MSE. Furthermore, analyzing two real datasets representing bank customer waiting times and daily wind speed data, MOM and WLSM demonstrate superior performance.

Keywords: Gamma distribution; Monte Carlo Simulation; Parameter Estimation Methods; Waiting times before service; Wind speed.

الملخص

التوزيع غاما مهم في مجالات الاحتمالات والإحصاء، ويُستخدم على نطاق واسع لنمذجة ظواهر متنوعة مثل أوقات الانتظار، وتوزيع الدخل، ومطالبات تعويضات التأمين، وسرعة الرياح. في هذه الدراسة، استكشفنا دقة أربعة أساليب لتقدير المعلمات الإحصائية، هي طريقة الترجيح الأعظم (MLM)، وطريقة العزوم (MOM) ، طريقة المربعات الصغرى (LSM)، و طريقة المربعات الصغرى الموزونة (WLSM) في تقدير معلمات الشكل والمقياس لتوزيع غاما من خلال محاكاة مونتي كارلو عبرعينات ذات حجم متنوع ومختلف القيم لمعلمات الشكل و السلم. أظهرت نتائج محاكاة مونتي كارلو أن طريقة MLM تميل إلى تقدير معامل الشكل بشكل منحاز موجب، بينما تعتبر LSM و WLSM الأكثر عدم انحيازًا لمعامل الشكل. فيما يتعلق بمعامل السلم، تفوقت MOM في تقديره مع تحيز وسيط قريب من الصفر و MSE قيمة وسيطه منخفضة. علاوة على ذلك، عند تحليل مجموعتين من البيانات الفعلية تمثلان أوقات انتظار عملاء البنك وبيانات سرعة الرياح اليومية، أظهرت MOM و WLSM أداءًا متفوقًا.

[*] Corresponding author: b_taoussi@esc-alger.dz

1. Introduction

The gamma distribution holds significant importance in the fields of probability and statistics, gaining substantial attention in literature. Notably, its association with the exponential distribution made the gamma distribution as a prevalent probability model for various waiting time scenarios [1]–[3]. Gamma distribution is also related the chi-square distribution and consequently the chi-square test. The latter is widely used in inferential statistics. For example, it is used in the goodness-of-fit of certain distribution in modeling data, determining the independence of two categorical variables in contingency tables, and also to assess the independence of responses in surveys [4], [5].

In queuing theory, Lee and Wang (2011) [3] modeled the waiting times of customers arrive randomly and served by a single server where the customers may leave the queue if their services do not commence before some conditioned time. Various skewed probability distributions such as exponential, lognormal, and Weibull distributions were compared to the gamma distribution. However, the latter was proved to perform satisfactorily using a simulation study. Lin and Lin (2015) [2] predicted the waiting times of a mobile ticket dispenser system to enhance user experience in the ordering process. Using the gamma distribution to model the waiting times, the authors showed that the mobile ticket dispenser system can effectively assist the ordering process. They simulated a scenario of popular restaurant that is crowded, with large waiting queue length, and the cook prepares meals with good management.

In economics, the gamma distribution finds vital applications. The study of income distribution is crucial for understanding the economic health of a society and promoting long-term stability and growth. Salem and Mount (1974) [6] approximated the personal income per family in the United States using the gamma distribution. They found that the gamma demonstrated better fit the data than the lognormal distribution. Moreover, they stated that the two parameters of a gamma density can be directly related to measures of inequality and of proportionate growth. The appropriateness of the gamma distribution in describing the characteristics of the distribution of income was further demonstrated in this review article [7]. In reference [8], the authors address the problematic issue of accurately evaluating income distribution and its impact on poverty and hunger. They critique existing indicators, such as the Gini coefficient and the share of the population in absolute poverty, in representing the distribution of people suffering from insufficient income. The authors propose a new method using the Gamma distribution to estimate income distribution parameters, aiming for a more comprehensive indicator. They apply this method to analyze the connection between income distribution and food demand, providing a more thorough analysis of income inequality and its implications for poverty and hunger.

In non-life insurance, the frequency-severity model uses Poisson-Gamma generalized linear model to describe insurance claims to obtain the optimum premium pricing [9], [10]. The claim frequency examines the number of claims, whereas the average claim severity takes account of the average amount of claims. Poisson distribution is often used to describe the frequency of claims, and gamma distribution is deployed to model insurance severity [9].

The usage of the gamma distribution is not limited in the aforementioned fields. In the domain of quality and reliability engineering, the gamma distribution finds extensive application for fitting product lifetimes, demonstrating versatility in capturing decreasing, constant, and increasing failure rates [11]. Beyond this, the gamma distribution has proven to be a fitting model in diverse fields such as environmental studies, wireless communications, geoscience, disaster monitoring, and image analysis [1]. Moreover, It has been used in the field of wind energy to describe the stochastic behavior of wind speed data [12].

After establishing the versatility application of the gamma probability distribution, the next consideration is how to effectively utilize it for modeling various types of data, such as those related to economics, insurance, wind speed, and other fields. Specifically, given that functionality of gamma distribution can be hindered if the scale and shape parameters are poorly estimated, the question arises: What estimation technique should be employed to ensure the accurate estimation of gamma parameters?

Statistical parameter estimation methods play a drastic role in un-tapping the full potential of a probability distribution to correctly model data. Maximum Likelihood Method (MLM) stands out as a widely used and reliable technique. It enjoys desirable mathematical and optimality features [13], [14]. An alternative to MLM is the Least Squared Method (LSM) is an alternative to MLM, yet it has less optimum desirable characteristics than the latter [15]. On the other hand, the Method of Moments (MOM) is a simpler technique that doesn't necessitate analytical methods for estimating parameters.

The objective of this study is to investigate the accuracy of different methods in estimating the scale and shape parameters of the gamma probability distribution. To evaluate the performance of these parameter estimation methods, we conducted a Monte-Carlo simulation study, considering diverse sample sizes and varying values of the gamma parameters. The primary contribution of this research is twofold: firstly, Monte Carlo simulation studies offer a structured and controlled environment for the comparison and evaluation of various estimation methods. Secondly, assist researchers in making informed decisions regarding the most appropriate approach for estimating the gamma parameters regarding the characteristics of their data. The rest of this study is organized as follows: the next section explains the probabilistic characteristics of the gamma distribution. Section 3 outlines the parameter estimation methods. Section 4 shows the results of the Monte-Carlo simulation analysis, and the final section provides a summary of the key findings.

2. Probabilistic characteristics of the Exponential distribution

Karl Pearson introduced the gamma distribution in 1895 [16]. It includes the exponential as a special case and can be very skewed, to being almost a bell-shaped density [17]. It arises naturally as the density of the sum of a number of independent exponential random variables [17]. Let X be a non-negative continuous random. X is said to have a gamma distribution with a shape parameter β and scale parameter α if its probability density function (PDF) is expressed as [18]

$$f_X(x) = \frac{x^{\beta-1}}{\alpha^{\beta}\Gamma(\beta)} e^{-\frac{x}{\alpha}}; x \geq 0, \beta > 0, \alpha > 0 \quad (1)$$

Where $\Gamma(a)$ is the gamma function $\Gamma(a) = \int_0^{\infty} t^{a-1} e^{-t} dt$.

The cumulative distribution function (CDF) In the case of the gamma distribution is not available in closed form. given as [17]

$$F_X(x) = P(X \leq x) = \int_0^x \frac{1}{\alpha^{\beta}\Gamma(\beta)} t^{\beta-1} e^{-\frac{t}{\alpha}} dt = \frac{1}{\Gamma(\beta)} \gamma(\beta, x/\alpha) \quad (2)$$

Where $\gamma(a, x)$ is the incomplete lower gamma function $\gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt$.

The PDF and CDF of the exponential distribution with shape parameter $\beta \in \{1, 2, 3\}$ and $\alpha \in \{2,4, 6\}$ are illustrated in Figure 1. The distribution is similar to the Erlang when β is a positive integer. Moreover, if β is less than or equal to one, the distribution takes on an exponential-like form with a mode at zero; however, for β exceeding one, the mode shifts to a value greater than zero. Notably, as β rises, the distribution gradually assumes a shape like the Gaussian distribution [16].

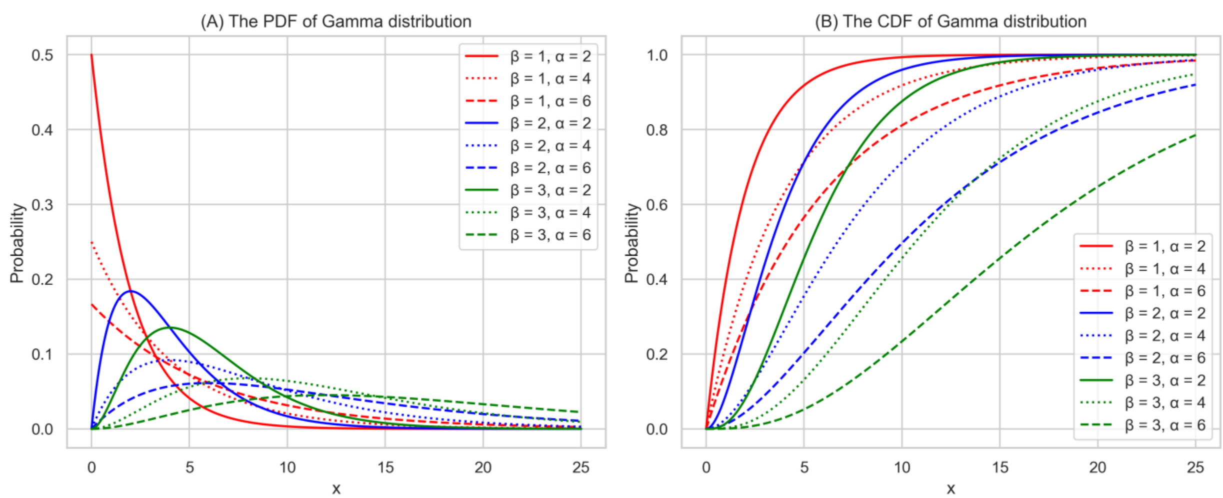

Figure 1 the PDF and CDF of the exponential distribution

The r-th raw moment of the gamma distribution is given as

$$E(X^r) = \alpha^r \frac{\Gamma(r+\beta)}{\Gamma(\beta)} \tag{3}$$

Table 1 shows the probabilistic characteristics of the Gamma distribution. This include the mean $E(X)$, the mode M_o, the variance $V(X)$, the coefficient of variation $CV(X)$, skewness Sk, and excess kurtosis Kr [17]. It should be noted that the median does not have readily calculable formula, because the CDF is not available in closed form.

Table 1 Probabilistic characteristics of the Gamma distribution

Measures	Statistics	values
Central tendency	$E(X)$	$\alpha\beta$
	M_o	$\alpha(\beta-1)$ if $\beta \geq 1$
Spread	$V(X)$	$\alpha^2\beta$
	$CV(X)$	$1/\sqrt{\beta}$
Asymmetry and Tailedness	Sk	$2/\sqrt{\beta}$
	Kr	$6/\beta$

The mean, median, and mode of the gamma distribution are plotted in Figure 2. These statistics increase monotonically as the shape and/or scale increases. It's worth mentioning that the mean consistently surpasses the median, indicating a positive skewness in the distribution.

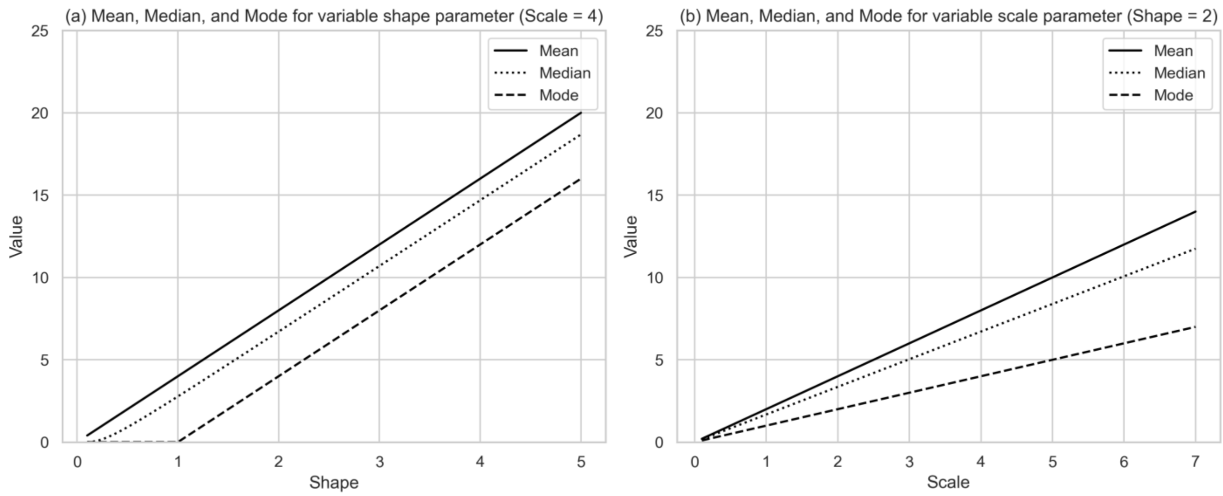

Figure 2 Mean, Median, and Mode of the gamma distribution

The coefficient of variation, skewness, and kurtosis, assessing the variability, asymmetry, and tail behavior of the gamma distribution in these respect, are only dependent to the shape parameter as outlined in Table 1. Figure 3 visually presents these statistics. It is evident from Figure 3 that an increase in the shape parameter leads to a decrease in variability, asymmetry, and tail behavior. Notably, outliers in the gamma distribution are observed to the right of the peak. Additionally, the distribution exhibits longer tails compared to the normal distribution, as indicated by its kurtosis.

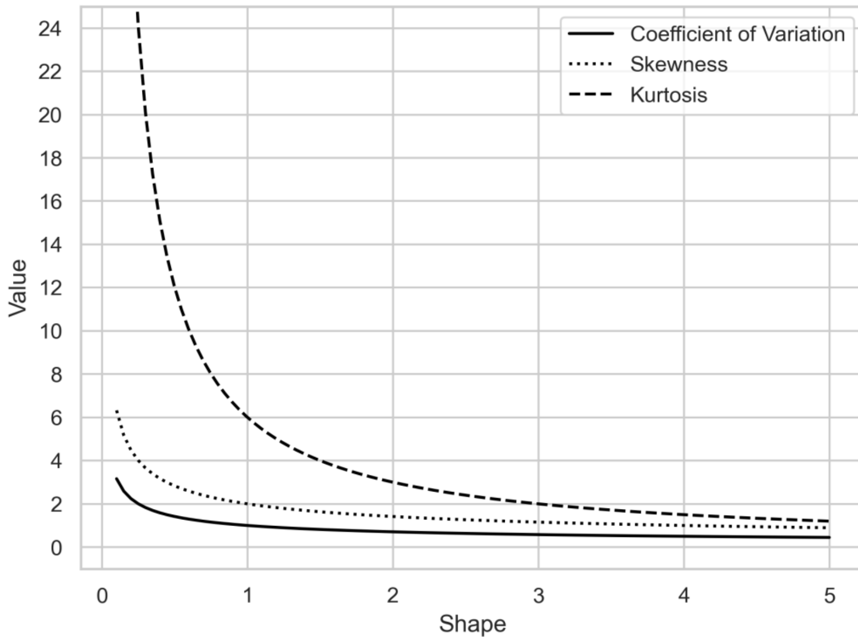

Figure 3 Coefficient of variation, skewness, kurtosis of the gamma distribution

3. Parameter estimation methods

In this section, the maximum likelihood method (MLM), the method of moments (MOM), the least-squares method (LSM), and a variate LSM called the weighted least-squares method (LSM) are discussed.

3.1 Maximum likelihood method

MLM is widely used point estimation method [14]. Its primary attraction arises from the ability to derive some highly general properties, including efficiency, impartial, asymptotic consistency and normality of invariance [10], [19]. The fundamental idea behind maximum likelihood estimation is that a good choice for the estimates $(\hat{\alpha}, \hat{\beta})$ of the parameters of interest (α, β) is the value of the parameter that makes the observed data most likely to have occurred $L(x_i; \alpha, \beta)$. This function is called the "likelihood function", and it is essentially the product of sampling densities for each observation in the sample. To simplify the procedure, we derivative the logarithm of the likelihood function [20].

In the case of the gamma distribution, the likelihood function is

$$L(x_i; \lambda) = \prod_{i=1}^{n} \frac{{x_i}^{\beta-1}}{\alpha^{\beta}\Gamma(\beta)} e^{-\frac{x_i}{\alpha}}$$

$$L(x_i;\lambda) = (\alpha^{\beta}\Gamma(\beta))^{-n}\prod_{i=1}^{n} x_i^{\beta-1}\, e^{-\sum_{i=1}^{n}\frac{x_i}{\alpha}} \tag{4}$$

Therefore the likelihood function is

$$LL(x_i;\lambda) = -n\ \ln\Gamma(\beta) - n\beta\ln\alpha + (\beta-1)\sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n}\frac{x_i}{\alpha} \tag{5}$$

The partial derivatives according to α and β respectively are:

$$\frac{\partial LL(x_i;\alpha)}{\partial\alpha} = \frac{-n\beta}{\alpha} + \frac{\sum_{i=1}^{n} x_i}{\alpha^2} \tag{6}$$

$$\frac{\partial LL(x_i;\beta)}{\partial\beta} = n(\Psi(\beta) + \ln\alpha) - \sum_{i=1}^{n}\ln(x_i) \tag{7}$$

$\Psi(\beta) = \frac{\acute{\Gamma}(\beta)}{\Gamma(\beta)}$ is the digamma function. Equating (6) to zero yields the maximum likelihood estimate of the scale parameter ($\hat{\alpha}_{MLM}$)

$$\hat{\alpha}_{MLM} = \frac{\bar{x}}{\hat{\beta}_{MLM}} \tag{9}$$

Where $\bar{x}$ is the sample mean. Replacing (9) in (7) yields

$$\frac{\partial LL(x_i;\beta)}{\partial\beta} = n(\Psi(\beta) + \ln\bar{x} - \ln\beta) - \sum_{i=1}^{n}\ln(x_i) \tag{10}$$

Analytically closed expressions for the MLM estimate of the shape parameter ($\hat{\beta}_{MLM}$) cannot be obtained and the solution of the equation $\frac{\partial LL(x_i;\beta)}{\partial\beta} = 0$ requires analytical techniques such as Newton-Raphson method [21].

3.2 The method moments

MOM is the oldest point estimation method. While it may not exhibit optimal properties compared to MLM, yet it is widely used due to its simplicity, efficiency in computational labor, and the fact that it can be improved upon easily in certain cases [19]. To obtain MOM estimates we equate the population raw moments to their peers from the sample. Since the gamma distribution has two parameters, the MOM estimates are found by solving this system of equation

$$\begin{cases} E(X) = \frac{\sum_{i=1}^{n} x_i}{n} \\ E(X^2) = \frac{\sum_{i=1}^{n} x_i^2}{n} \end{cases} \tag{11}$$

$$\begin{cases} \alpha\beta = \frac{\sum_{i=1}^{n} x_i}{n} \\ \alpha^2\beta = \frac{\sum_{i=1}^{n} {x_i}^2}{n} \end{cases} \tag{12}$$

3.3 Least squares method

the least-squares estimates are found by minimizing the sum of the squares of the deviations of the cumulative distribution function $F(x)$ from the empirical cumulative distribution function $\hat{F}(x_{(i)})$ [22].

$$argmin_{\hat{\alpha}_{LSM},\hat{\beta}_{LSM}} \sum_{i=1}^{n} \left[F(x_{(i)}) - \hat{F}(x_{(i)})\right]^2 \tag{13}$$

Where

$$\hat{F}\left(x_{(i)}\right) = \frac{i-0.5}{n+1} \tag{14}$$

3.4 Weighted Least squares method

WLSM was developed as an extension of LSM to tackle specific constraints and issues encountered in regression analysis. LSM is susceptible to the influence of extreme outliers. Therefore, by assigning lower weights to outliers, WLSM can mitigate their impact on parameter estimates. Expression (13) becomes [23] Weighted Least squares

$$argmin_{\hat{\lambda}_{WLSM}} \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{(i)}) - \hat{F}(x_{(i)})\right]^2 \tag{15}$$

4. Results and discussion

The Monte-Carlo simulation study aims to investigate the performance of the maximum likelihood method, the method of moments, the least-squares method, and the method of weighted least-squares in estimating the parameters of the gamma distribution. The study was conducted using the programming language Python 3.9 under the following settings:

- The sample sizes are $n = 50, 200, 500, 1000, 5000, 10000$.
- The number of iterations is taken $1000000/n$.
- The values of the shape parameter $\beta = 1, 2, 3$.
- The values of the scale parameter $\alpha = 2, 4, 6$.

Bias and MSE were used to evaluate the quality of the estimates provided by MLM, MOM, LSM, and WLSM. For an unknown parameter θ, Bias is a measure of how much the estimated values $\hat{\theta}$ deviated from the true values θ. A good estimate has bias close to. It is expressed as

$$Bias\left(\hat{\theta}\right) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta_i\right) \quad (18)$$

MSE, on the other hand, is a measure of variability in the estimates. It indicates how stable the estimates are. Low MSE implies that the estimate is efficient. MSE is calculated as follows

$$MSE\left(\hat{\theta}\right) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta_i\right)^2 \quad (19)$$

4.1 Analysis of simulation results for the shape parameter

Figure 4 depicts the bias in the shape parameter of the Monte Carlo simulation employing MLM, MOM, LSM, and WLSM techniques. Notably, these methods mostly exhibited negative bias, in exception for MLM, suggesting a tendency to underestimate the shape parameter. Furthermore, they generally exhibited convergence with increasing sample sizes. Figure 5 presents the MSE for the shape parameter in the Monte Carlo simulation results, revealing a trend of lower MSE with larger sample sizes for most of these methods. The following observations were also made:

- For $\beta = 1$, all methods exhibit negative bias, with LSM having values closest to zero, followed by WLSM, MLM, and MOM in this order. Regarding MSE, MLM stands out as the most stable method with the lowest MSE, closely followed by WLSM and LSM. Despite the scale parameter varying between 2 and 6, its impact on bias and MSE appears to be minimal. Both bias and MSE continue to decrease with increasing sample size. In summary, MOM is the least suitable method for estimating β, yielding large biased estimates with high variance. It is noteworthy that when $\beta = 1$, the gamma distribution reduces to the exponential distribution with parameter $\lambda = \alpha$.
- For $\beta = 2$, all methods exhibit negative bias, except for MLM, indicating that it tends to overestimate β. Notably, the bias and MSE of MLM fail to converge as n increases, making it as the least appropriate method in this scenario. In contrast, LSM once again excels in terms of bias and low variance.

- When $\beta = 3$, LSM produces the most accurate estimates. It is observed that as β increases, both bais and MSE also increase changing from the range [-0.1, 0] and [0, 0.08] for $\beta = 1$ to [-0.25, 0.1] and [0, 0.6] for $\beta = 3$.

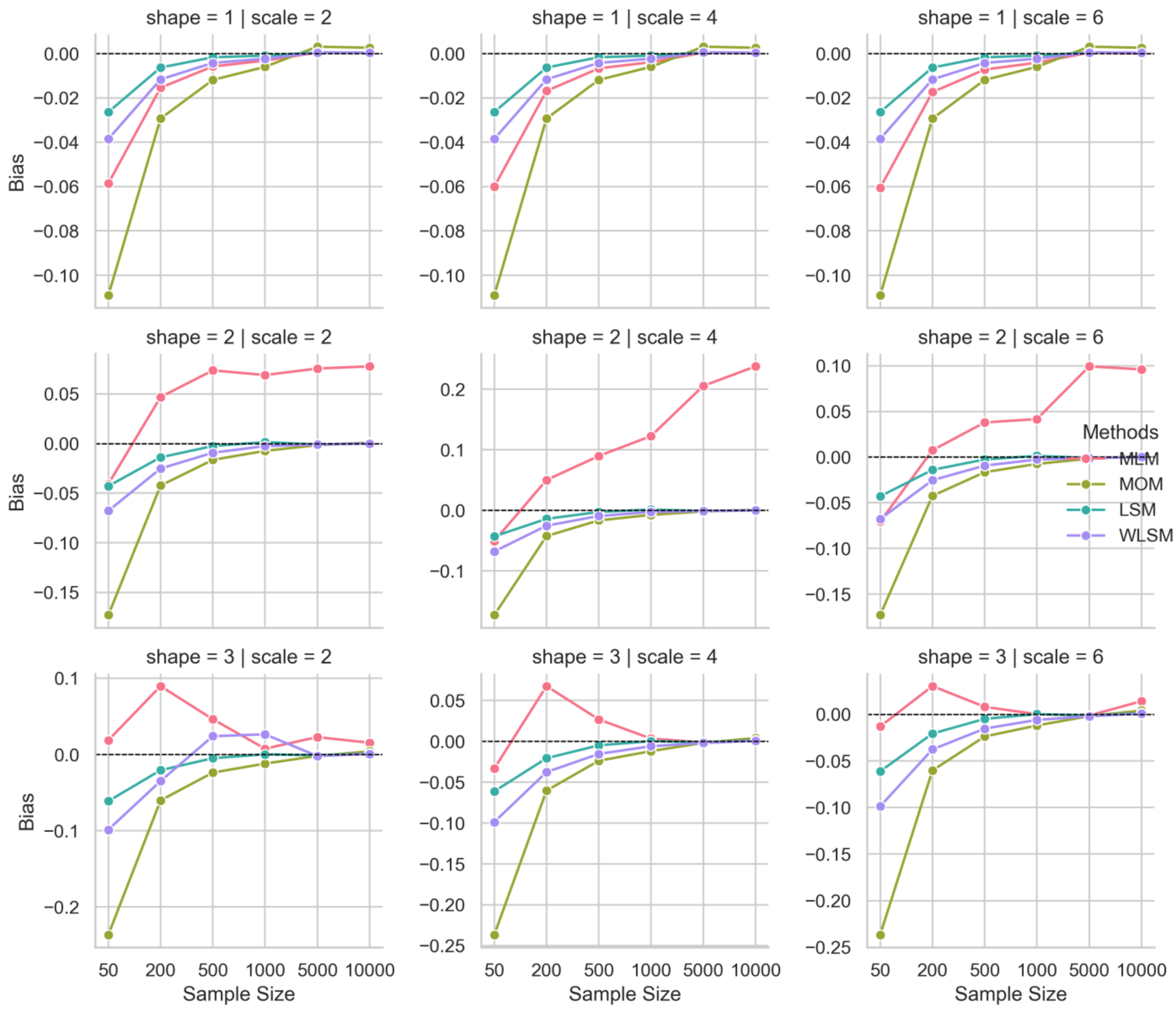

Figure 4 Bias of the shape parameter for the gamma distribution

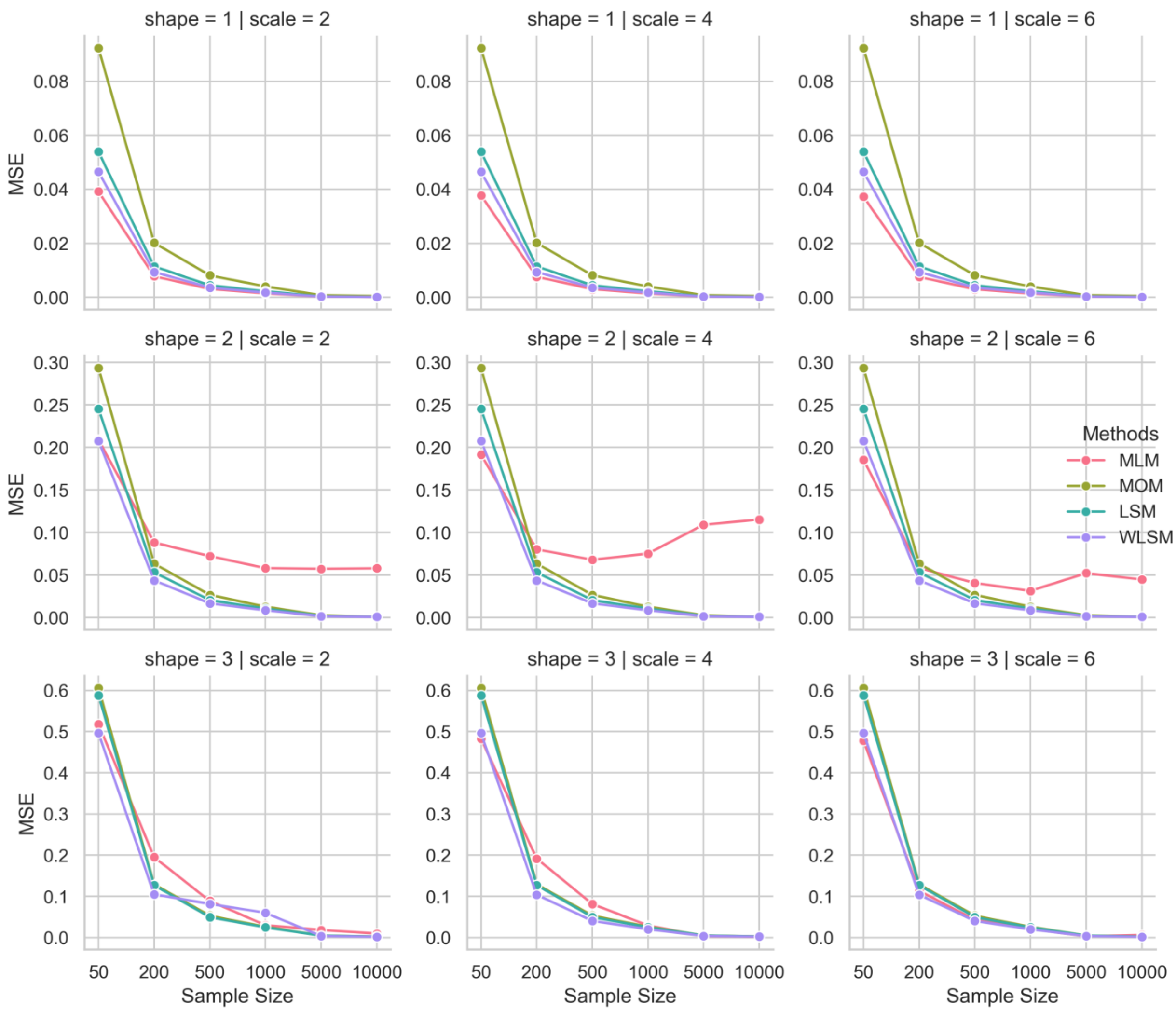

Figure 5 MSE of the shape parameter for the gamma distribution

4.2 Analysis of simulation results for the scale parameter

The following observations were made about the scale parameter:

- For $\alpha = 2$, the methods fluctuates between overestimating (MLM and MOM) and underestimating (LSM, WLSM) the scale parameter. However, it seems that MLM encounters challenges in convergence when $\beta = 2$, even with increasing sample size.

- For $\alpha > 2$, similar trends to the previous case are observed. However, the magnitude of Bias and MSE rises with rising scale parameter. It appears advisable to not use MLM for estimating the gamma distribution when $\beta = 2$.

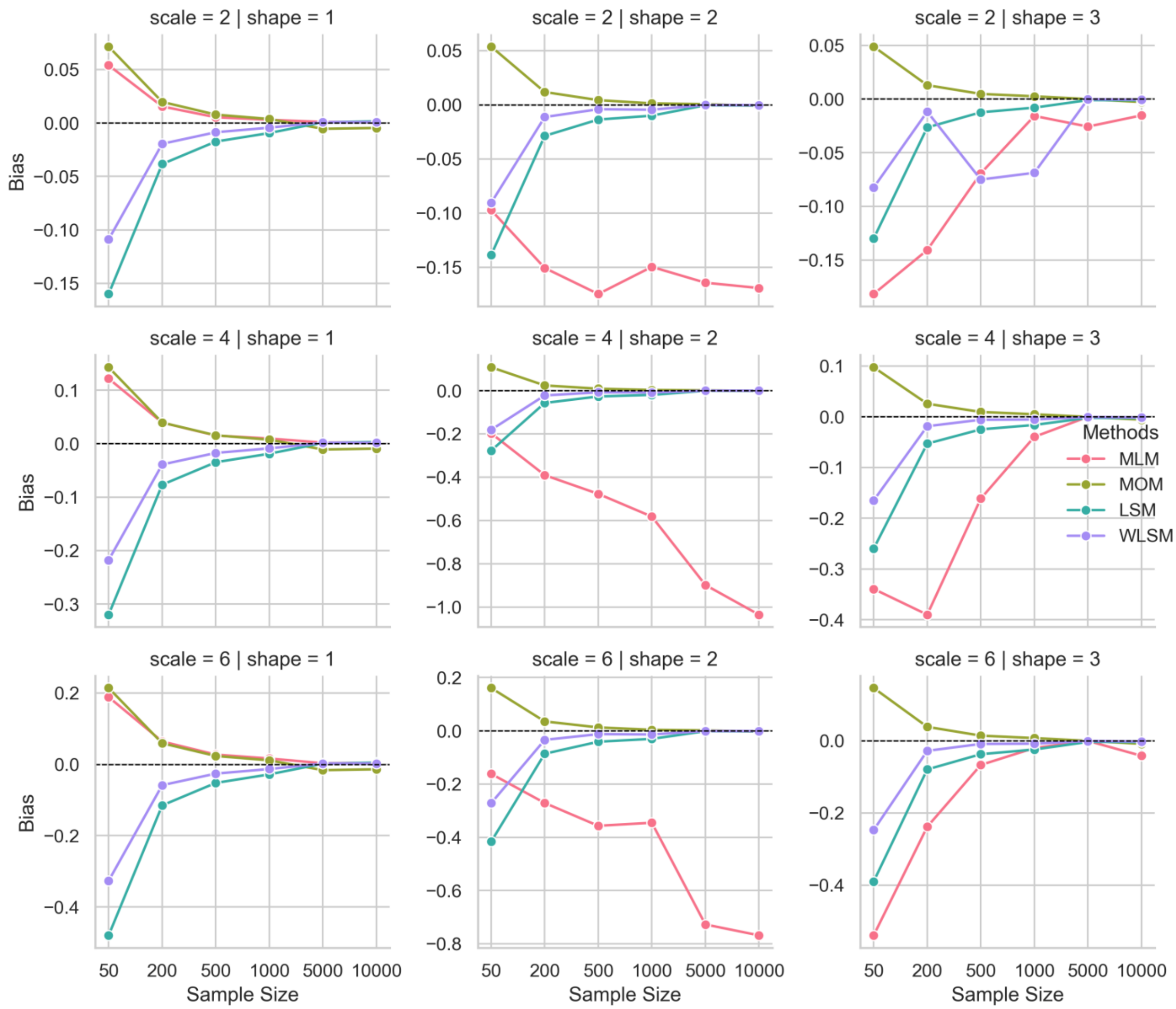

Figure 6 Bias of the scale parameter for the gamma distribution

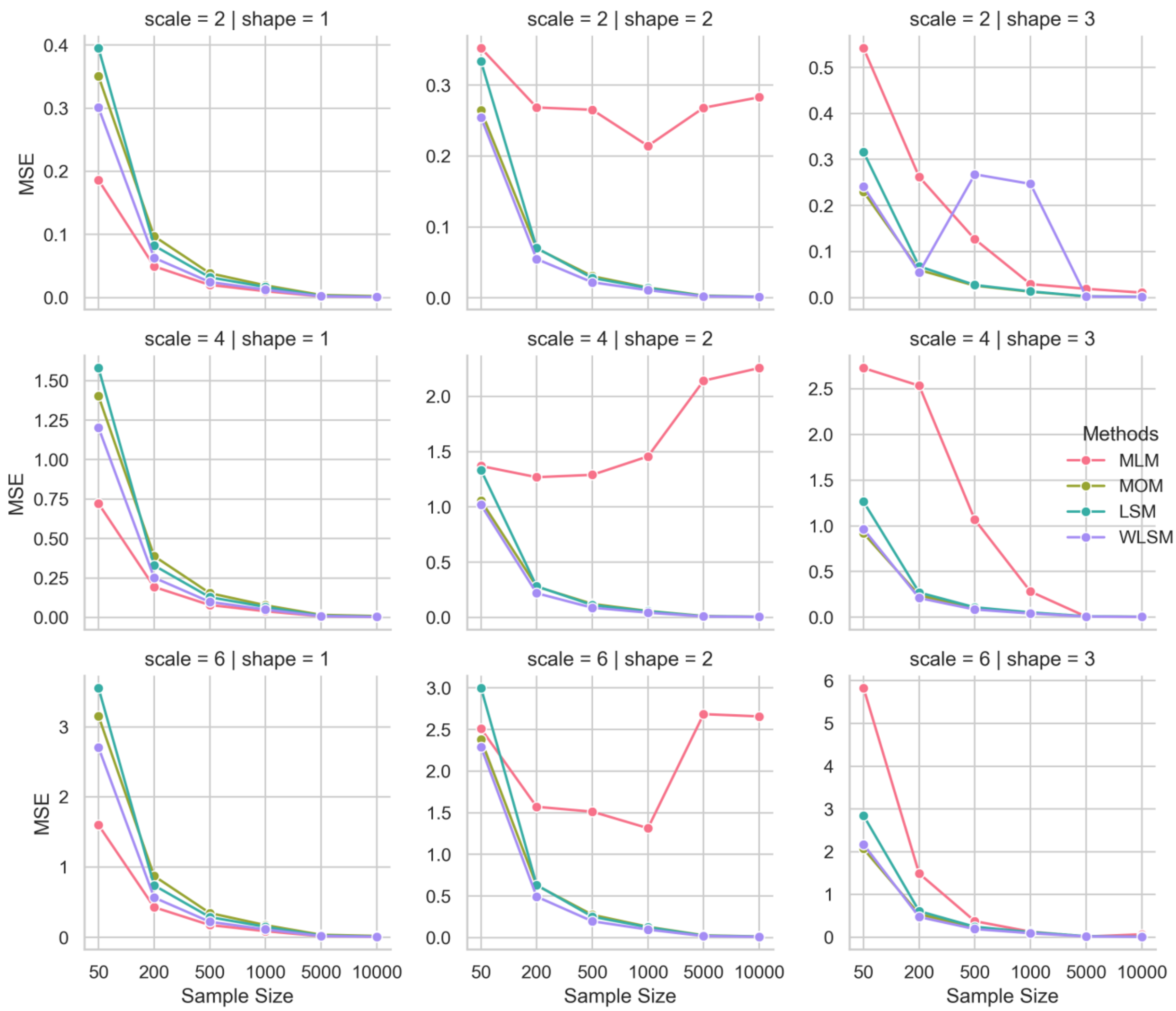

Figure 7 MSE of the shape parameter for the gamma distribution

4.3 Analysis of the overall performance

The results of the Monte Carlo simulation study across all the values of shape and scale parameters, and the sample sizes are combined and illustrated as boxplots in Figure 8. The upper two plots showcase the bias of MLM, MOM, LSM, and WLSM estimates, whereas the bottom two plots represent their MSE. Similar to the prior observations, only MLM tends to overestimate the shape parameter and exhibits the largest median MSE. Conversely, LSM emerges as the most unbiased method for estimating the shape parameter. However, MOM outperforms LSM in estimating the scale parameter, displaying the closest median bias to zero and moderately low median MSE, followed by WLSM.

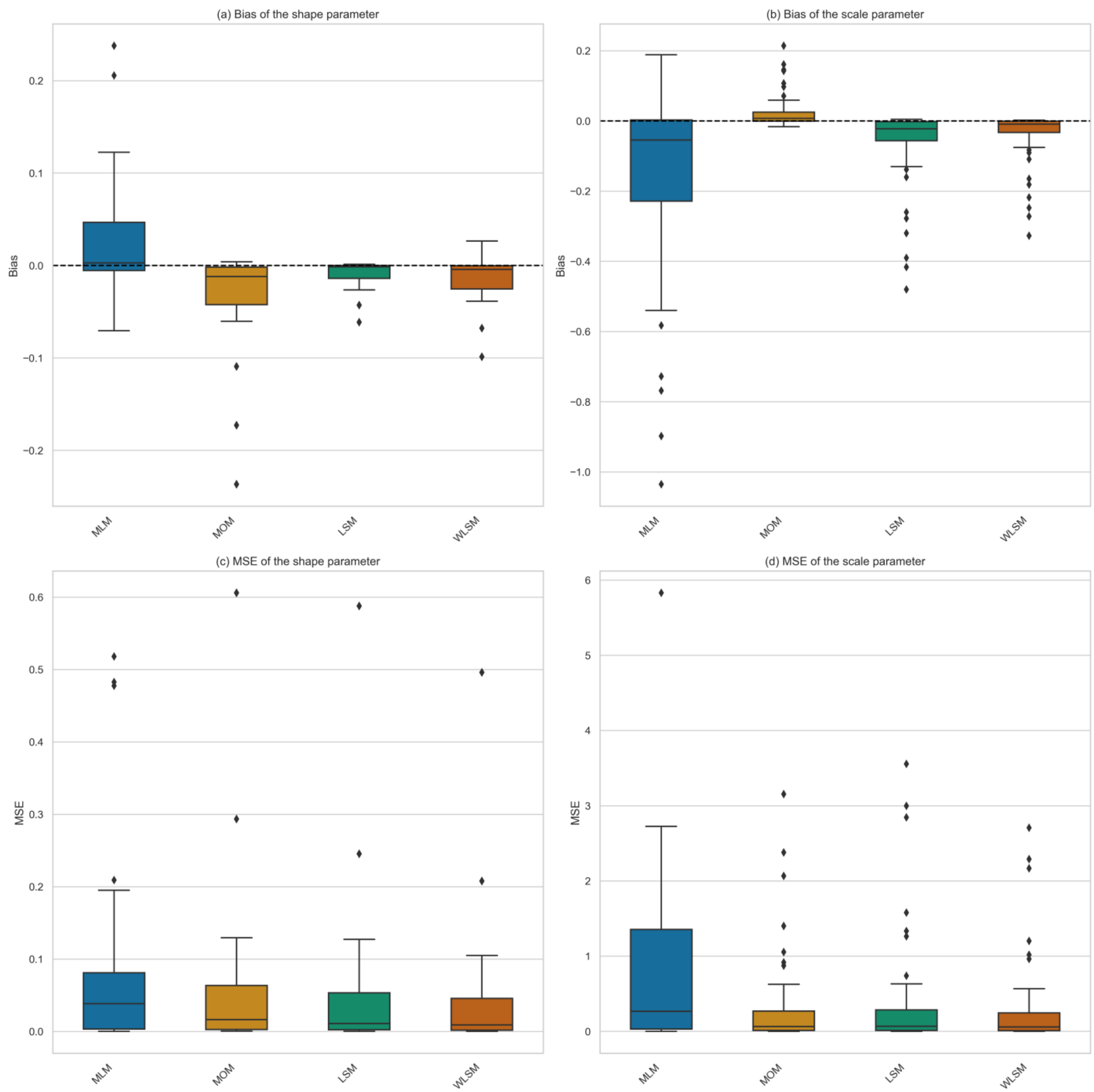

Figure 8 comparative analysis of estimating methods for the gamma distribution parameters

4.4 Data analysis

In this section, the parameter estimation techniques are tested on two datasets, waiting times before service and daily wind speed data. RMSE is used to evaluate the performance of MLM, MOM, LSM, and WLSM in estimating the shape and scale of the gamma distribution.

Dataset 1: This data represents the waiting times (in minutes) before service of 100 bank customers [23].

Table 2 Goodness-of-fit results of Dataset 1

Methods	MLM	MOM	LSM	WLSM
Shape	2.0088	1.8814	1.9449	1.9532
Scale	4.9168	5.2496	5.0641	5.0859
RMSE	0.0368	**0.0364**	0.0367	**0.0364**

It is noted from Table 2 that MOM and WLSM provide the smallest RMSE values, indicating relatively better goodness-of-fit compared to MLM and LSM. Therefore, MOM and WLSM may be preferred for this dataset based on the RMSE criterion.

Dataset 2: This data represents the daily wind speed data measured at 10 m above the ground level for the month of December between 2006 and 2010 at Djelfa (155 observations) [24].

Table 3 Goodness-of-fit results of Dataset 2

Methods	MLM	MOM	LSM	WLSM
Shape	2.1082	2.5103	1.7173	2.1909
Scale	2.0588	1.7291	2.7604	2.0437
RMSE	0.0200	0.0213	0.0209	**0.0194**

Table 3 indicates that WLSM has the smallest RMSE value, indicating the best goodness-of-fit for Dataset 2. It is followed closely by LSM. MLM and MOM have slightly higher RMSE values. Therefore, WLSM appears to be the most suitable method for this dataset based on RMSE.

5. Conclusion

The gamma distribution is a significant probability model in probability and statistics, commonly used for modeling waiting times, income distribution, insurance claims, wind speed, and various other fields. It is associated with the exponential and chi-square distributions, widely employed in inferential statistics, queuing theory, economics, and non-life insurance. In this study, the gamma distribution is explored for parameter estimation accuracy, with the maximum likelihood method (MLM), method of moments (MOM), least squared method (LSM), and weighted least squared method (LSM) compared. The research uses Monte-Carlo simulation studies to evaluate these methods across diverse sample sizes, aiming to guide researchers in selecting the most suitable approach for estimating gamma distribution parameters based on their data characteristics. In addition, Two datasets, representing bank customer waiting times and daily wind speed data, were subject of parameter estimation analysis using MLM, MOM, LSM, and WLSM, with goodness-of-fit assessed via RMSE. The following conclusions were drawn:

- MLM tends to overestimate the shape parameter with the largest median MSE, while LSM is the most unbiased for the shape parameter.
- MOM outperforms LSM in estimating the scale parameter with a median bias close to zero and moderately low median MSE.
- For Dataset 1, MOM and WLSM yield the smallest RMSE values, indicating better fit than MLM and LSM.
- Dataset 2 results show WLSM with the smallest RMSE, followed by LSM, making WLSM the most suitable method for Dataset 2 based on the RMSE criterion.

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