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Mathematical model for controlling thermal deformation in composite beam with sinusoidal fiber volume fraction

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Abstract

Many thin structural components such as beams, plates and shells experience a through – thickness temperature variation. This temperature variation can produce both an in-plane expansion and an out-of-plane (bending) curvature. For use in engineering structures, we often wish to minimize the thermal deformation of a component or to match it to the thermal deformation of another component. This is accomplished by using a composite whose fibers have a negative axial thermal expansion coefficient. By varying the fiber volume fraction within a symmetric laminated beam to create a functionally graded material (FGM), certain thermal deformations can be controlled or tailored. Specifically, a beam can be designed which does not curve under a steady – state through – thickness temperature variation. Continuous gradation of the fiber volume fraction in the FGM layer is modeled in the form of sinusoid function of the coordinate axis in thickness direction of the beam. The beam results are independent of the actual temperature values, within the limitations of steady – state heat transfer and constant material properties. The influence of volume fiber fraction distributions are studied to match or eliminate an in –plane expansion coefficient, or to match a desired axial stiffness. Combining two fiber types to create a hybrid FGM can offer desirable increase in axial and bending stiffness while still retaining the useful thermal deformation behavior.

Keywords: Composite materials, Fiber volume fraction, Thermal deformation

1. Introduction

With the development of the industries such as the aeronautics, astronautics, national defenses and nuclear energy, conventional homogeneous materials become more and more difficult to meet the higher and higher needs of these high-tech industries. In conventional configurations, flat beams and plates are made of plies, the fibers within each ply being parallel and uniformly spaced. However, it is possible that significant increases in structural efficiency may be obtained by varying the fiber spacing packing them closely together in regions where great stiffness is needed but less densely in other regions. Shiau and Lee [1] studied stress concentration around holes in composite laminates with variable fiber spacing. It proves that reducing the fiber volume ratio near the hole edges can significantly reduce the stress concentration in that region. Recently, Meftah et al. [2] considered the seismic analysis of reinforced concrete coupled shear walls structures strengthened by bonded composite plates having variable fibers spacing. Dynamic analysis was performed to investigate the influence of the fibers redistribution of the bonded plates on the lateral deflections of RC coupled shear walls. Meftah et al. [3] also presented a finite element model for static and free vibration analysis of reinforced concrete shear walls structures strengthened with thin composite plates having variable fibers spacing. Numerical results were obtained for six no uniform distributions of E-glass, graphite and boron fibers in epoxy matrices. The fiber redistributions of the bonded plates are seen to increase the frequencies modes and reduce substantially the lateral displacements. Kuo and Shiau [4] studied the buckling and vibration of composite laminated plates with variable fiber spacing by use of finite element method. Benatta et al [5] presented a mathematical solution for bending of short hybrid composite beams with variable fibers spacing. Bedjilili et al [6] presented exact solutions for shear flexible symmetric composite beams with a variable fiber volume fraction through thickness. Bouremana et al. [7] proposed a polynomial distribution of fiber volume fraction to control the thermal deformation of composite beam.

When a structure is subjected to a non-uniform temperature field, it normally reacts by producing deformations composed of (in-plane) expansion and (out-of-plane) bending. These deformations are usually undesirable since they distort the structure and cause stresses when parts expand unequally.

With the availability of materials such as graphite and Kevlar fibers which possess a negative axial thermal expansion coefficient and high stiffness, a composite may be made which exhibits a negative axial coefficient of thermal expansion (CTE). If we desire a structure with zero CTE in the plane, we can do this by combining layers with positive CTE with layers having negative CTE. We can also fabricate a composite whose fiber volume fraction (V_f) equals the critical value (V_c) so that the CTE is everywhere zero (although the resulting composite may have low and may thus have low stiffness). Zero CTE ensures dimensional stability and minimal thermal mismatch stresses amongst similar parts of a structure subject to a uniform temperature field. If we join zero CTE components with non – zero CTE components, however, there will be mismatch stresses and distortion even for a uniform temperature.

For the case of a non – uniform temperature field, the problem becomes more complex. In actual application, the structure usually experiences a temperature variation through its thickness as one side is exposed to sunlight, an impinging gas or liquid, or a heat source. The zero in – plane CTE laminated composite structures do not provide protection against this temperature field. If they are in laminated form, they will bend, and the bending deformation can be considerable. A beam with uniform volume fraction, (the critical value), can provide protection against bending, but will have an in – plane CTE of zero (may not match neighboring structures) and may be insufficiently because of low V_f.

Wetherhold et al. [8, 9] have proposed design methods to control both thermal deformation and mismatches in thermal deformation. One method is to bond dissimilar composite or composite and metal layers together [8]. This creates potential manufacturing difficulties concerning the layer interfaces. If it is not possible to co – cure the beam, then a separate surface preparation and bonding step will be needed. The other method is to use functionally graded materials (FGM) in the form of composites whose volume fraction varies through the thickness (according to linear or parabolic variation) to avoid the difficulties of bonding dissimilar layers together [9]. In the present paper, we use FGM beams with variation of volume fraction of fibers based on sinusoid law.

2. Mathematical model

2.1. Continuous variation of fiber volume fraction

Consider a symmetric beam of thickness 2h (Fig.1). The fiber volume fraction, vary continuously through the thickness of beam (in z direction) as follows:

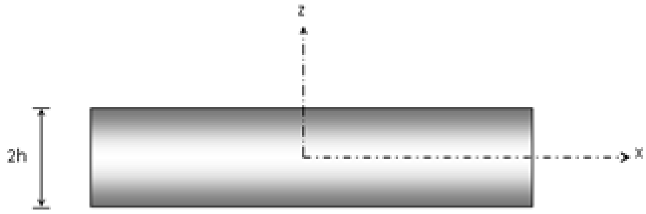

Figure 1 .Symmetric functionally graded beam.

$$V_f(z) = V_{av} - B\cos\left(\frac{\pi z}{h}\right) \tag{1}$$

Where V_{av} is the average value of fiber volume fraction, and B is the sinusoid amplitude defining variation of material properties through the thickness of beam. The stiffness and thermal expansion coefficient for a uniaxial reinforcement are given by:

$$E(z) = E_f V_f(z) + E_m\left(1 - V_f(z)\right) \tag{1b}$$

$$\boldsymbol{\alpha}(z) = \frac{\boldsymbol{\alpha}_f E_f V_f(z) + \boldsymbol{\alpha}_m E_m\left(1 - V_f(z)\right)}{E(z)} \tag{2b}$$

Where subscripts f and m refer to fiber and matrix, respectively. Equations (2) are simple rule – of – mixtures approximations which are accurate for unidirectional, continuous – fiber composites [10].

To describe the deformation of the symmetric beam, we evaluate the thermal and mechanical force and moment resultants and relate them, through laminate stiffness's, to the bending curvature and in – plane strain. This also allows us to calculate the CTE of the beam.

The mid – plane strain, $\boldsymbol{\varepsilon}_x^0$, and the bending curvature, k_x, are related to applied mechanical force, N_x, and moment, M_x, resultants and thermal force, N_x^T, and moment, M_x^T, resultants by:

$$N_x = A\boldsymbol{\varepsilon}_x^0 - N_x^T \tag{3}$$

$$M_x = Dk_x - M_x^T \tag{4}$$

The beam has in – plane stiffness, A, and out – of – plane or bending stiffness, D, given by:

$$A = \int_{-h}^{h} E(z)\, d\, z > 0 \quad (5a)$$

$$D = \int_{-h}^{h} E(z)\, z^2 d\, z > 0 \quad (5b)$$

The mechanical and thermal force and moment resultants are defined by:

$$N_x = \int_{-h}^{h} \sigma_x dx \quad (6a)$$

$$N_x^T = \int_{-h}^{h} E(z)\, \alpha(z)\, T(z)\, d\, z \quad (6b)$$

$$M_x = \int_{-h}^{h} \sigma_x z\, d\, z \quad (7a)$$

$$M_x^T = \int_{-h}^{h} E(z)\, \alpha(z)\, T(z)\, z\, d\, z \quad (7b)$$

Consider the case of zero applied mechanical loading ($N_x = M_x = 0$) to obtain the mid – surface thermally induced strain, ε_x^0, and the thermal curvature, k_x. Equations (3) and (4) will reduce and can be inverted to give:

$$\varepsilon_x^0 = \frac{1}{A} N_x^T \quad (8)$$

$$k_x = \frac{1}{D} M_x^T \quad (9)$$

The effective laminate in – plane CTE (α_x) can be found by simplifying Eq (8) for the case of a uniform temperature field:

$$\alpha_x \equiv \frac{\varepsilon_x^0}{\Delta T} = \frac{1}{A} \int_{-h}^{h} E(z)\, \alpha(z)\, d\, z \quad (10)$$

The temperature profile, $T(z)$, is linear through the thickness if the transverse thermal conductivity, k, is constant. If k was a strong function of V_f, the temperature profile would deviate from linearity and the beam design would have to be 'tuned' for a specific temperature profile. In fact, k is a weak function of $V_f(z)$ for a polymer matrix with typical fibers [11]. It has been shown that the importance of k is distinctly secondary to that of E and α [12], and k is assumed constant in this paper. The temperature profile is thus:

$$T(z) = T_1 + \frac{(T_2 - T_1)}{2h}(z + h) \quad (11)$$

Where (T_1, T_2) is the temperature with $T_1 = T(z = -h)$ and $T_2 = T(z = h)$.

The properties of the fibers and the epoxy matrix used in this paper are given in Table 1.

Table1

Fiber and matrix properties

	AS-Gr	Kevlar	P100-Gr	Epoxy
E (GPa)	277.67	137.76	772.0	3.5
α (10^{-6}/°C)	-0.969	-5.49	-1.40	65
V_c (%)	50.8	23.1	17.4	–

For each fiber, there is a critical volume fraction, V_c, at which the longitudinal CTE is zero (see Eq(2b)). A composite with $V_f = V_c$ experiences neither in – plane nor bending expansion under a through – thickness temperature variation. It should be noted that fiber CTE values increase with increasing temperature. This analysis is thus only applicable over a temperature range in which the fiber CTE is roughly constant and negative.

Using Eqs (1) and (2) to evaluate the integrals of Eqs (5) – (10), we obtain the normalized thermal moment, $\overline{M}^T$, and beam CTE, $\bar{\alpha}$, as:

$$\overline{M}^T = \frac{3\pi^2 M_x^T}{h^2 E_m \alpha_m (T_2 - T_1)} = (f\, g - 1)(V_{av}\pi^2 + 6B) + \pi^2 \quad (12a)$$

And

$$\bar{\alpha} \equiv \frac{\alpha_x}{\alpha_m} = \frac{(f\, g - 1)V_{av} + 1}{(f - 1)V_{av} + 1} \quad (12b)$$

Where the ratios f and g are given by $f = E_f / E_m > 0$ and $g = \alpha_f / \alpha_m < 0$.

With no applied mechanical loads, we achieve zero bending curvature ($k_x = 0$) when $\overline{M}^T = 0$; see Eq (4). Equations (12) can be used to determine the values of V_{av}

and B which will eliminate thermal curvature, as well as give a prescribed CTE for the composite. This is illustrated in case 1 and case 2 below. Note from Eq (12a) that the results are independent of the actual temperature profile, as long as it is steady – state. This can be shown by considering the heat transfer rate, $\dot{q} = -k(T_2 - T_1)/2h$ through the beam. Setting $\overline{M}^T = 0$ is equivalent to setting. $f_1(V_{av}, B) = (f\ g - 1)(V_{av}\pi^2 + 6B) + \pi^2 = 0$

Setting $\overline{M}^T = 0$ **is** equivalent to $(h^2 E_m \alpha_m (T_2 - T_1)/3\pi^2) f_1(V_{av}, B) = 0$ which can also be expressed as setting $(-2\dot{q}h^3 E_m \alpha_m / 3\pi^2 k) f_1(V_{av}, B) = 0$

All of these conditions are identical.

2.1.1 Case 1

Enforce the condition $k_x = 0$ (i.e. $\overline{M}^T = 0$). There are two unknowns, V_{av} and B, and one condition. We may set the value of V_{av} and solve for B; the resulting value of $\overline{\alpha}$ is then found (see Table 2).

Table 2
Results for specified values of V_{av}, $k_x = 0$

Materials	V_{av}(%)	B (%)	$\overline{\alpha}$ (10^{-3})	$\overline{E} = E_x / E_m$
AS Gr/Ep	40	17,71	8	26,6
	50,8	-0,05	0	33,5
	60	-15,18	-5	39,4
Kevlar/Ep	15	13,36	5,2	6,7
	23,1	0,04	0	9,8
	30	-11,31	-2,4	12,5
P100 Gr/Ep	10	12,15	1,8	23,0
	17,4	-0,02	0	39,2
	30	-20,74	-1,1	66,9

In figures 2 – 4, we present the variation of V_f across the beam thickness for different values of V_{av} and for different used materials (AS Gr/Ep; Kevlar/Ep; P100 Gr/Ep). The presented variation of V_f, leads to the desired condition $k_x = 0$ (i.e. $\overline{M}^T = 0$). Note that the Kevlar and P100 fibers are very effective in controlling the deformation of the matrix. The very low V_f values required may give a composite whose stiffness is too low; this will be addressed in Section 2.2.

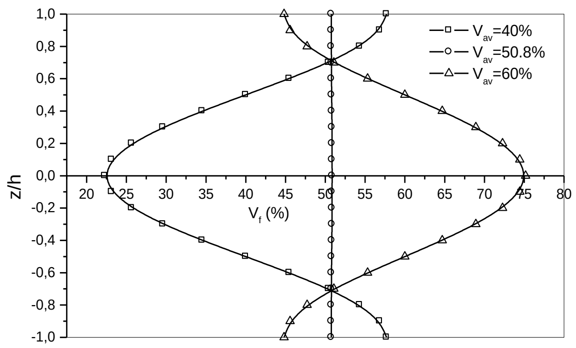

Figure 2. Fiber volume fraction V_f (%) across the beam thickness for AS Gr/Ep material

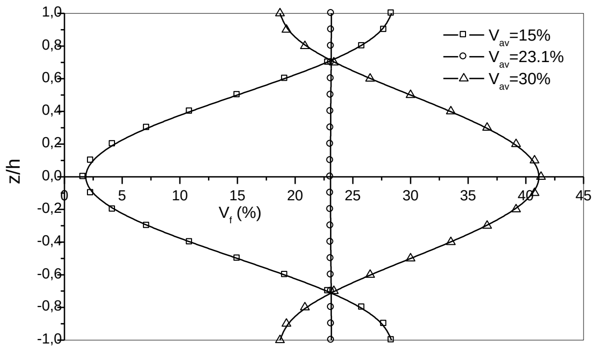

Figure 3. Fiber volume fraction V_f (%) across the beam thickness for Kevlar/Ep material

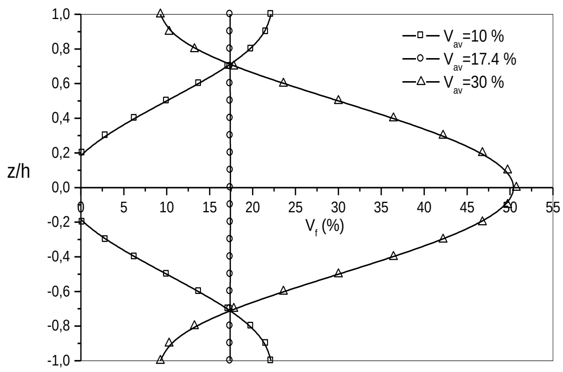

Figure 4. Fiber volume fraction V_f (%) across the beam thickness for P100 Gr/Ep material

2.1.2 Case 2

For this case, we set two conditions: no thermal curvature ($k_x = 0$) and a specified value of $\overline{\alpha}$. Values of

V_{av} and B result from these conditions. Figures 5 – 7 show the variation of V_{av} and B for different specified values of $\bar{\alpha}$ and for different used materials (AS Gr/Ep; Kevlar/Ep; P100 Gr/Ep). It can be seen from these results that V_{av} and B vary almost linearly with $\bar{\alpha}$. However, the sinusoid amplitude B increases with $\bar{\alpha}$, contrary to the average value of fiber volume fraction V_{av}.

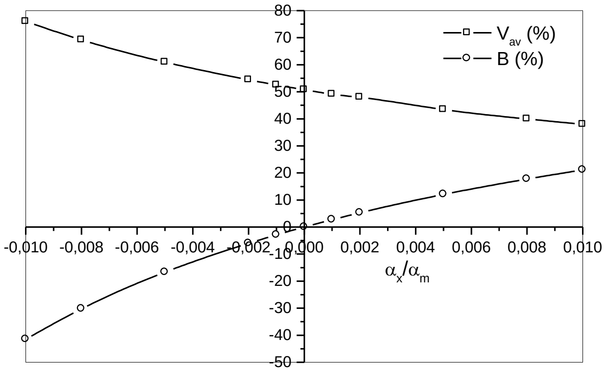

Figure 5. Average values of fiber volume fraction V_{av} (%) and the sinusoid amplitude B (%) versus the specified values of $\bar{\alpha}$ for AS Gr/Ep material.

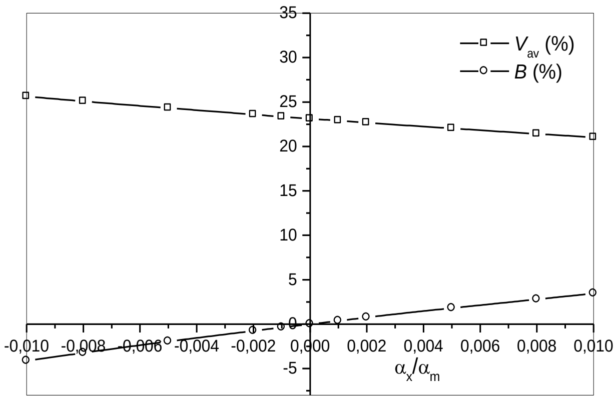

Figure 6. Average values of fiber volume fraction V_{av} (%) and the sinusoid amplitude B (%) versus the specified values of $\bar{\alpha}$ for Kevlar/Ep material.

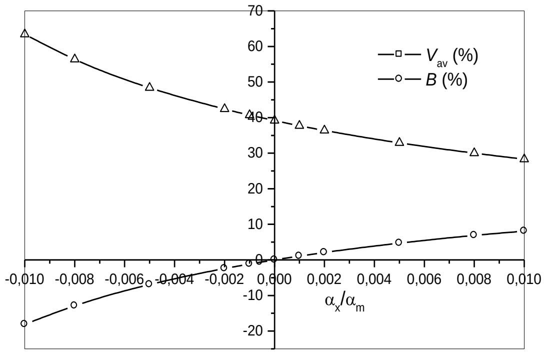

Figure 7. Average values of fiber volume fraction V_{av} (%) and the sinusoid amplitude B (%) versus the specified values of $\bar{\alpha}$ for P100 Gr/Ep material.

2.2. HYBRID COMPOSITES

As seen in figures 2 – 7, the volume fractions needed to satisfy the thermal deformation requirements can be low. This may lead to composites which do not possess sufficient stiffness for the design requirements. It is possible to create a hybrid which combines fibers with a positive CTE (usually glass) with graphite or Kevlar varies in *z* as before. A similar set of effective axial properties can be found:

$$E(z) = E_f V_f(z) + E_g V_g + E_m \left(1 - V_f(z) - V_g\right) \qquad (13a)$$

$$\alpha(z) = \frac{\alpha_f E_f V_f(z) + \alpha_g E_g V_g + \alpha_m E_m \left(1 - V_f(z) - V_g\right)}{E(z)} \qquad (13b)$$

Where subscripts f, g and m stand for graphite or Kevlar, glass and matrix, respectively. The critical volume fraction of graphite or Kevlar to obtain CTE = 0 will increase as more glass fiber is added. Properties of glass are: $E = 72\,\text{GPa}$, $\alpha = 5.10^{-6}\ {}^{\circ}\text{C}^{-1}$ [12].

Normalized forms for the thermal moment, CTE, and effective in – plane CTE are given by:

$$\bar{M}^T = \frac{3\pi^2 M_x^T}{\alpha_m E_m (T_2 - T_1) h^2} = (f\ g - 1)\ (V_{av}\pi^2 + 6B) + (d\ j - 1)\ V_g + \pi^2 \qquad (14a)$$

$$\bar{\alpha} = \frac{\alpha_x}{\alpha_m} = \frac{(f\ g - 1)V_{av} + (d\ j - 1)V_g + 1}{(f - 1)V_{av} + (d - 1)V_g + 1} \qquad (14b)$$

$$\bar{E} = \frac{E_x}{E_m} = (f - 1)V_{av} + (d - 1)V_g + 1 \qquad (14c)$$

Where $f = E_f / E_m > 0$, $g = \alpha_f / \alpha_m < 0$, $d = E_g / E_m > 0$, $j = \alpha_g / \alpha_m > 0$.

To illustrate the design of a beam with hybrid reinforcement, we set $k_x = 0$ and specifying the values of $\overline{\alpha}$ and $\overline{E}$.

The volume fraction of glass fiber V_g at specified values of $\overline{\alpha}$ and $\overline{E}$ is presented in Table 3. As can be seen from the result, the volume fraction of glass fiber V_g increases with $\overline{\alpha}$.

Table 3
Volume fraction of glass fiber for hybrid composites, $k_x = 0$, $\overline{E} = 15$ and $\overline{\alpha}$ specified

Material	$\overline{\alpha}$	V_g (%)	V_{av} (%)	B (%)
vlar/Ep	-0.036	1.37	35.79	-20.54
	-0.024	7.82	32.5	-13.69
	-0.012	14.28	29.21	-6.84
	0	20.73	25.92	0
	0.012	27.19	22.62	6.84
	0.024	33.64	19.33	13.69
	0.036	40.1	16.04	20.54

3. Conclusion

As seen from this study, the availability of stiff fibers with a negative thermal expansion coefficient (α) creates interesting design possibilities for controlling thermal deformation. We may also wish to design so as to match an in – plane CTE or modulus. By creating FGM in the form of symmetric composite whose fiber volume fraction varies in the form of sinusoid function of the coordinate axis in thickness direction of the beam, we accomplish these design tasks. The results are independent of the actual temperature profile, given constant constituent material properties and steady – state heat transfer.

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