COST

Communication Science & Technologie N° 2 . Décembre 2003

Wavelet Analysis Versus Fourier Analysis : Application To Time-Varying Spectra Signals

S. A. Chouakri, F. Bereksi-Reguig***

(*) Laboratoire LTTNS Département d'Electronique Université de Sidi Bel Abbes BP 89 22000

e-mail: sa_chouakri@hotmail.com

(**) Département d'Electronique Université de Tlemcen 13000

ABSTRACT

We present in this paper an approach to understanding one of the recent tools used in the signal processing world, that is the Wavelet theory. The philosophy of this paper is based on utilizing the Fourier theory as a platform to illustrate the Wavelet analysis. For this purpose, we present the relationships between these two techniques, as well as the results of applying these techniques to three different cases representing signals with particular features. The obtained results show obviously the improved ability of the wavelet transform to analyze some complicated cases of signal regarded to the Fourier analysis.

RESUME

On présente dans cet article une approche permettant la saisie d'un outil récent utilisé en traitement du signal qui est la théorie des ondelettes. La phiplosophie de cet article est basée sur l'utilisation de la théorie de Fourier comme une plate forme pour la compréhension de la théorie des ondelettes. Dans ce contexte, on présente les relations entre ces deux théories ainsi que le résultat de leurs applications aux trois cas de signaux chacun avec des caractéristiques particulières. Les résultats obtenus montrent clairement la capacité supérieure de la transformée d'ondelettes dans l'analyse des cas compliquées de signaux vis à vis à la transformée de Fourier.

KEYWORDS

Wavelet transform, Fourier transform, Time-frequency resolution, Non-stationary signal, Time-varying spectrum

1- INTRODUCTION

One of the most important purposes of the signal processing is to extract the main features of the signal such as : peaks, transients, abrupt changes, frequency information… These objectives can be achieved into different manners by using different signal transformations or representations. This variety of transformations are imposed by the nature of the information looked for. A common way to represent a signal (usually time-varying) is to decompose it into elementary blocks as follows : $f= \sum_i f_i$ or more precisely $f =<\propto_i , \varphi_i >$ where φ_i are the transform windows or supports and $\propto_i$ are called the transform coefficients. This operation is called the synthesis or the reconstruction of the signal f. Conversely, the signal f can be represented using the transform coefficients following the reversed path that is : $\propto_i=<\Psi_I , f>$ where Ψ_i are the dual transform windows. This operation is well known as the analysis or the decomposition of the signal f.

The root work of the signal transformation theory is the Fourier transform. It was at the beginning a simple theory stating that any periodic signal can be decomposed to a set of elementary periodic signals. This theory has been exploited later and extended to the aperiodic signals and recovering now an arsenal of tools such as the DFT and FFT techniques. This technique is still a powerful tool in the signal processing field , whereas it is not so suitable in all the cases such as the non-stationary or the time-varying spectrum signals and those representing the transients and discontinuities; the problems that suffer from the most of the real signals : seismic, speech, biomedical…

An interesting extension of this technique was developed by Gabor in the 1940's known as the Short Time Fourier Transform. He introduced a sliding fixed size window and to translate it along the time axis with a fixed shift factor (τ). The idea was to scan the temporal occurrence of the signal events, the new concepts were risen, the time-frequency analysis of the signal and consequently the time-frequency resolution. One major limitation suffers from this technique is the fixed time-frequency resolution due to the fixed window size and shift values.

To drawback this limitation the wavelet theory, a mathematical concept developed by Haar in the early of the 10's, was introduced. The key idea was to use of a set of basic functions that are the scaled and shifted versions of an original limited duration function called the "mother" wavelet.
The wavelet analysis has been mainly used for the analysis of the non-stationary signals and those representing transients such as : drifts, sudden peaks, the rapid rising and falling edges of the events, trends, sudden changes, and discontinuities. Its development comprises several works of researchers and scientists in different fields as the applied mathematics, geophysics, physics, computer science, and engineering. In sum, the main support of the wavelet concept was the great work oh Haar in 1910's [1]. Basically, the term "wavelet" was first used in the 70's when J. Morlet improved the STFT performances by introducing a scaled (dilated/compressed) windows instead of the fixed size ones. Later, with his friend A. Grossman deepen their researches and found many applications of this new technique. In the middle of 80's Meyer with his collaborators came up with the orthornormal wavelet bases. Not far from, Lémarie and Battle came up with, independently, constructions of wavelet bases consisting of spline functions, with better decay than Meyer's wavelets, in price of some regularity [1]. Latter, Mallat and Meyer introduced the multiresolution analysis concept [2]. Since 1987, the idea of using the FIR filters for constructing orthornormal wavelet bases has led to the subband filtering. The application of this technique has introduced an undesirable aliasing effect, that has been cancelled by the use of the Quadrature Mirror Filters (QMF). One of the most known meticulous work for the construction of wavelet filters are that of Ingrid Daubechies [3] in late 80's and early 90's. Different researchers have contributed at the development of the wavelet concept and applications..

2- FOURIER ANALYSIS

Fourier theory is one of the most well known techniques used in signal processing world. Signal analysts have at their disposal an arsenal of tools : Fourier series decomposition, continuous and discrete Fourier transforms , the fast Fourier transform...
The Fourier analysis can be viewed from two sides :

- It decomposes any given time-varying signal x(t) into elementary building blocks $\varphi_i(t)$ and can be expressed as :

$x(t)=\Sigma\, c_i(t)\, \varphi_i(t)$ (1)

Where $\varphi_i(t)=e^{j2\pi ft}$ and $c_i(t)$ are called Fourier coefficients

- It transforms the time representation of the signal –x(t)- to its corresponding frequency representation (spectrum) X(f) and can expressed as :

$X(f)= \int x(t).\, e^{-j2\pi ft}\, dt$ (2)

We are interested in this paper to the continuous Fourier transform rather than the Fourier series decomposition.
The signal analysts know perfectly the Fourier theories and their consequences, what we will do in this paper is to denote some interesting remarks that be useful to illustrate the wavelet theory.
A brief study of the continuous Fourier transform shows some limitations that are:

- The FT is computed along an infinite interval of time domain;
- Reconstructing the original time-varying signal depends mainly on the cancellation of the high frequency Fourier coefficients, that is sensitive to high-frequency noise;
- The FT is perfectly local in frequency whereas it is global in time. It can localize any two very adjacent frequencies, while the time occurrence of these frequency components is completely lost. This limitation arises especially when dealing with the non-stationary signal.

This last limitation obligated the signal analysts to bring up with a new technique that allows the time localization of the analyzed signal. Gabor was the first to improve the continuous Fourier transform (CFT) by suggesting the short time Fourier transform (STFT) in the 40's [4]. He introduced a fixed size (duration) window –g(t)- that is translated along the time axis with a fixed shift factor -τ-. The non-stationary time varying signal is, then, divided into a sequence of segments in which the signal is considered to be quasi-stationary. The resulting transform is two-dimensional representation and it represents the time-frequency of the signal. The STFT is given by :

$STFT(f, \tau) = \int f(t).g^*(t-\tau).\exp(-j.f.t)\, dt$ (3)

This technique permits the time localization (ΔT) of the signal, while at the other hand the frequency localization (ΔF) is degraded due to the convolution operation of the signal spectrum with the window spectrum.
The time and frequency resolutions (localization's) of the analyzed signal are defined respectively as :

$$\Delta t^2 := \frac{\int t^2 \left|g(t)\right|^2 dt}{\int \left|g(t)\right|^2 dt} \qquad (4)$$

$$\Delta f^2 := \frac{\int f^2 \left|G(f)\right|^2 df}{\int \left|(f)\right|^2 df} \qquad (5)$$

These two parameters (ΔT and ΔF) depend firmly on the window and its spectrum sizes; they mean that a signal can not be represented as point in the time-frequency plan; its position can be only determined within a rectangle of ΔT*ΔF. More precisely, it means that it is impossible to discriminate between two adjacent frequencies components (f0 and f1) if they are not Δf apart each other (f1≥f0+Δf); similarly for the temporal case. Unfortunately these parameters can not be chosen independently since they are governed by what is called the uncertainty principle (or Heisenberg inequality) stating that : $\Delta T*\Delta F \geq 1/(4\pi) \cong 0.08$. The only solution is to trade time resolution for the frequency resolution or vice versa.

3- WAVELET ANALYSIS

The wavelet analysis includes an arsenal of approaches such as : continuous wavelet transform, wavelet series decomposition, discrete wavelet transform, fast wavelet transform algorithm... We limit our discussion, in this paper, on the continuous wavelet transform.

3.1-Continous Wavelet transform (CWT) :

As we have seen in the previous paragraph, the Fourier transform provides two basic approaches : the continuous Fourier transform and the STFT. The two parameters can be extracted from these two techniques respectively : frequency (dilatation / compression) of the trigonometric functions and the shift (translation) of the window. Also, we have remarked that the time-frequency resolution is fixed along the time-frequency plan.

Figure 1 shows, respectively, the time-based, frequency-based, and the STFT views of a given signal.

To overcome the limitation of the STFT, what is obviously needed is a transform view of a signal that has a changed sizes (Δt and Δf) of the information cells , of course respecting that $\Delta t*\Delta f \cong 0.08$, similar to the next figure (figure 2).

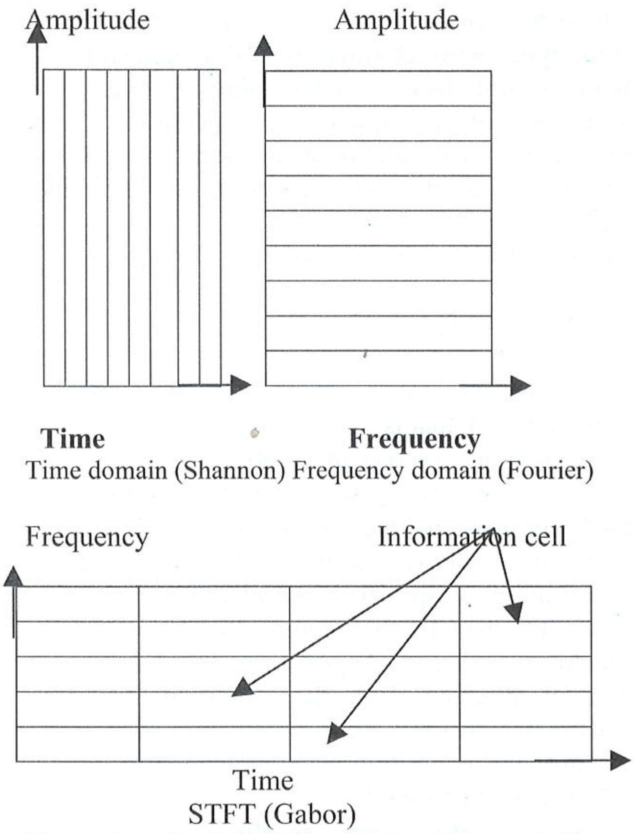

Figure 1 : time-view, frequency-view, and time-frequency view of the signal

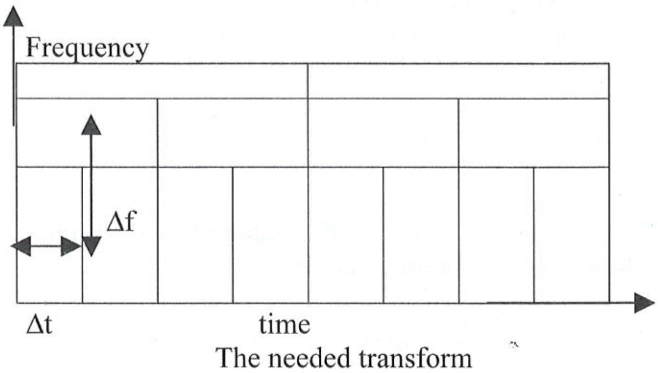

Figure 2 : The needed transform view.

This concept can be accomplished by introducing a scaled window (dilating / compressing) with a changed shift parameter along the time axis. We can imagine, so, a transform given by :

$Trsf(s,\tau,x(t)) = \langle x(t), \xi_{s,\tau}(t)\rangle$

The transform $Trsf(s,\tau,x(t))$ is the inner product of the signal x(t) with the scaled and shifted version of the original window ξ(t) by scaling and shifting factors (s,τ) respectively.

Indeed, the wavelet transform is based on this technique. The CWT is a set of coefficients computed by the inner product of the time-varying signal x(t) and a family of wavelets.

The continuous Wavelet transform CWT is given by: $CWT_{a,b}(f)=\langle f,h_{a,b}\rangle=\int x(t).\ h^*_{a,b}(t)\ dt$ (6)

where $h_{a,b}$ (t) are generated from a "mother" or 'prototype" wavelet –h(t)- by scaling and shifting with a shifting factor b and a scaling factor a.
The CWT of one-dimensional function f(t) is a two-dimensional function of the scale –a- and the shift-b- and is refereed to as the time-scale joint representation.
The wavelets $h^*_{a,b}$ (t) are given by:

$$h_{a,b}(t) := \frac{1}{\sqrt{\ }}\, h\left(\frac{t-b}{a}\right) \qquad (7)$$

The constant $(a)^{-1/2}$ guarantees the energy normalization, that is :

$$\int |h_{a,b}(t)|^2 dt = \int |h(t)|^2 dt \qquad (8)$$

So all the wavelets $h_{a,b}$ (t) of the same family have the same shape and the same energy.
This last remark leads us to state that the WT does not require a specified wavelet functions; in contrast to the CFT that requires the trigonometric functions as bases. Anyone can build his own wavelet function with the condition that the constructed wavelet must satisfy two properties that are : the admissibility and the regularity.

3.2- Wavelet properties

- **The admissibility condition**

The admissibility condition includes, in essence, the energy conservation in the time-scale space, that implies a possible reconstruction of the original signal x(t) from its CWT, and is given by:

$$(t) = \frac{1}{C\psi} \int\int CWT(a,b,x)\, h_{a,b}(t) \frac{1}{a^2}\, da\, db \qquad (9)$$

the reconstruction can be accomplished if the following condition is satisfied :

$$C\psi := \int \frac{|H(w)|^2}{w} dw < \infty \qquad (10)$$

where H(w) is the Fourier transform of the basic wavelet h(t).
the last condition requires that H(w) has to be zero at zero frequency value that is H(0)=0.
In other words, this implies that the average value of the wavelet function is zero that is : $\int h(t).dt=0$
This means that the wavelet function has to oscillate, that where it comes the world "wave".

- **The regularity condition**

The regularity condition is imposed to the wavelet function in order to make it local in frequency domain, furthermore the wavelet function is already local in time domain. For this purpose, the FT of the wavelet function is null at zero frequency, what is required now is to impose that H(w) has to decay and vanish above a certain frequency or scale value.
Mathematical calculus lead to that the speed of the convergence of the CWT coefficients towards zero with decreasing of the scale b (to limit the band pass of the filter H(w)) is determined by the order of the wavelet h(t) -N-, e.g. the first null moments of the order up to N of h(t) that is [2]:

$Mp=\int t^p h(t).dt=0$ for p=0, 1, …N (11)

In the frequency domain, this is equivalent to the Nth derivative of H(f) to be equal to zero at the zero frequency that is :

$H^{(p)}(0)=0$ for p=0, 1, …N

This equation implies that H(f) has N+1 zeros at the zero frequency ; so we can write H(f) as :

$$H(f)=f^{(N+1)} \sum_{i=0}^{k} a_i.f^i \qquad (12)$$

with $k \in Z$.
obviously, the H(f) vanishes faster and sharper as the order N of the wavelet increases.
These two properties (conditions) gave the studied function the term "Wave-Let".

3.3- Time-frequency resolution

As it has been stated, the CWT is defined as: $CWT_{a,b}(x)=\langle x(t),h_{a,b}(t)\rangle$
Using Parseval's identity, the CWT can be defined as :

$$CWT_{a,b}(x)=(1/2\pi) \langle x,H_{a,b}(w)\rangle \qquad (13)$$

where $H_{a,b}(w)$ is the FT of the $h_{a,b}(t)$ and can be defined as :

$$H_{a,b}(w) =(a)^{1/2}\, e^{-jwb} H(w) \qquad (14)$$

At the other hand, we have seen in the previous paragraph that the wavelet function h(t) and its spectrum are of finite band, we can then define their finite centers Tc and Fc and their radii ΔT and ΔF respectively (see figure 3).
These quantities are defined as [3, 6]:

$$Tc := \frac{1}{\|\ \|^2} \int t\, |h(t)|^2 dt \qquad (15)$$

$$\Delta T := \frac{1}{\|\ \|^2} \int (t-Tc)^2\, |h(t)|^2 dt \qquad (16)$$

and similarly for Fc and ΔF.

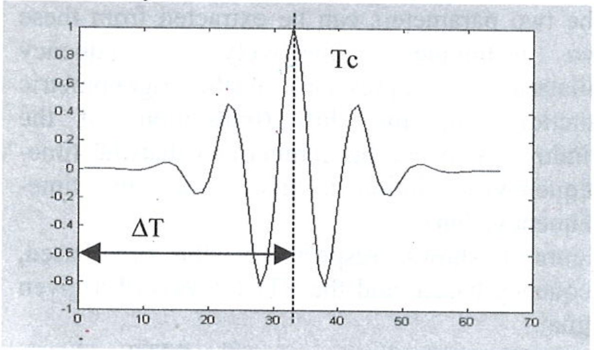

Figure 3: the center Tc and radix ΔT of the Morlet wavelet

It can proven that the CWT with a specified scale and shift factors (a0 and b0) picks up the information within the time interval [b0+a0.Tc-a0.ΔT , b0+a0.Tc+a0.ΔT] and the frequency interval [(Fc-ΔF)/a0 , (Fc+ΔF)/a0]. These two intervals determine the time-frequency window (or the information cell) sizes. It is proven, again, that the information cells sizes of the CWT are governed by the scale and the shift values. In summary, we can simulate the CWT to microscope where the scale and the shift values represent the zoom (in, out) and the position of the picked image respectively.

3.4- Discrete versions of the CWT

The CWT requires an infinite scanning of the signal x(t) along infinite points of scales and shifts, that is a critical problem arises when dealing with numerical systems. A trivial solution for this problem is to discretize these two parameters. One of the most commonly discetization method used is that based on the dyadic grid, that is:

$a=2^i$ and $b=k.2^i$ with i and $k \in Z$ [7].

The resulting WT using this form is known as the wavelet series decomposition (WSD). This remains continuous while time-scale parameters become discrete.

The discrete WT (DWT) is the wavelet transform of a discretized signal x(n) with discetized time-scale parameters. It is mainly related to the multiresolution analysis (MRA) concept.

3.5- Applications of the wavelet transform

The wavelet analysis is a powerful technique that has been exploited in different applications of the signal processing world. It is an efficient tool to analyze the non-stationary signals and those representing transients. It is applied to several types of signals : seismic, biomedical, speech, images...[1, 2, 8]. Generally, the CWT is suited to signal analysis , while its semi-discrete and full discrete versions (WSD and DWT) are mainly used for signal coding applications including multiresolution analysis, compression, coding-decoding...

4- RESULTS AND DISCUSSION

We have applied both the FT and CWT separately to some different types of signals, showing each a particular feature, in order to evaluate the efficiency of the CWT versus the FT.

We have chosen three cases of signals that are : the first one representing a discontinuity and a sharp transient; the second case represents an original signal and its shifted version in time; while the third case is a superposition of two linear chirp functions of the form $f(t)= \exp(j\pi\propto_1 t^2) + \exp(j\pi\propto_2 t^2)$

Figures 4, 5 ,and 6 show the results of applying the CFT and CWT to the three cases denoted respectively. In each figure (4, 5, and 6) we show three signals : the first one at the top (a) shows the temporal representation of the studied signal, the second one in the middle (b) shows the spectral representation of the signal using the FFT command [9], while the third one at the bottom (c) shows the time-scale analysis of the signals by applying the CWT command.

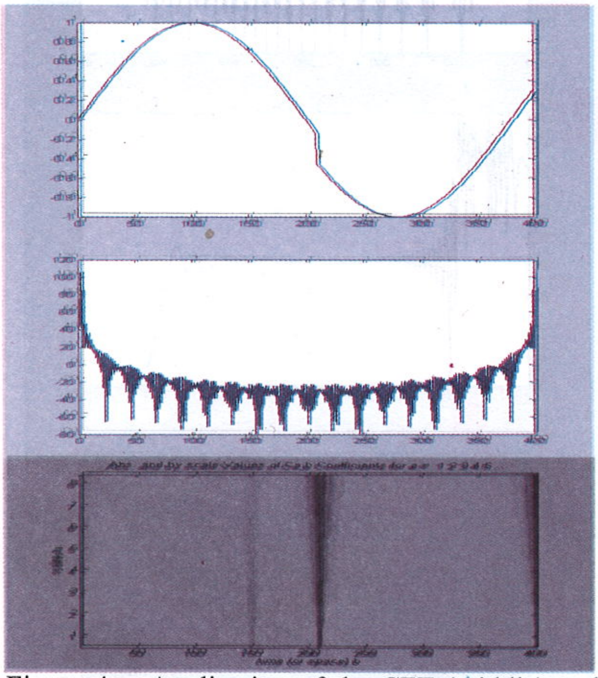

Figure 4 : Application of the CFT (middle) and CWT (bottom) to transient signal (top)

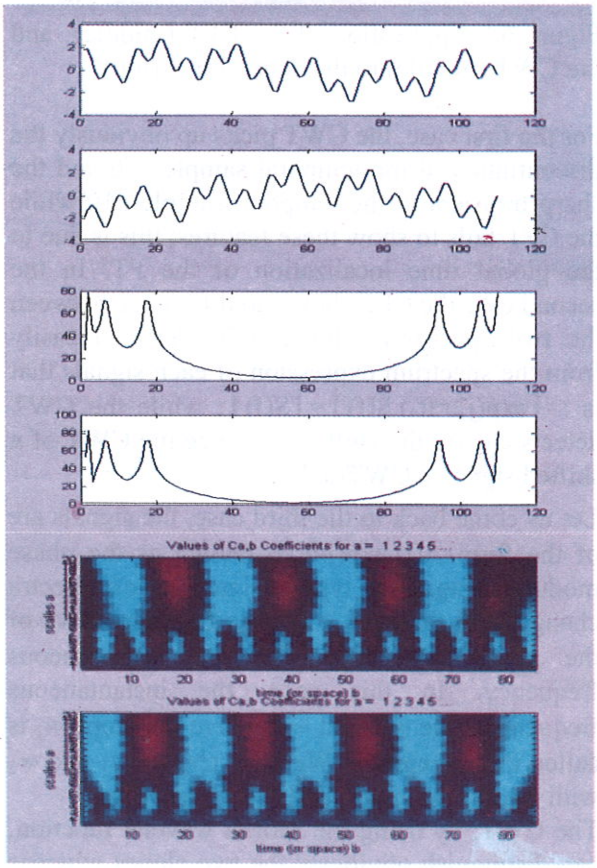

Figure 5 : Application of the CFT (middle) and the CWT (middle) to the shifted signal (top)

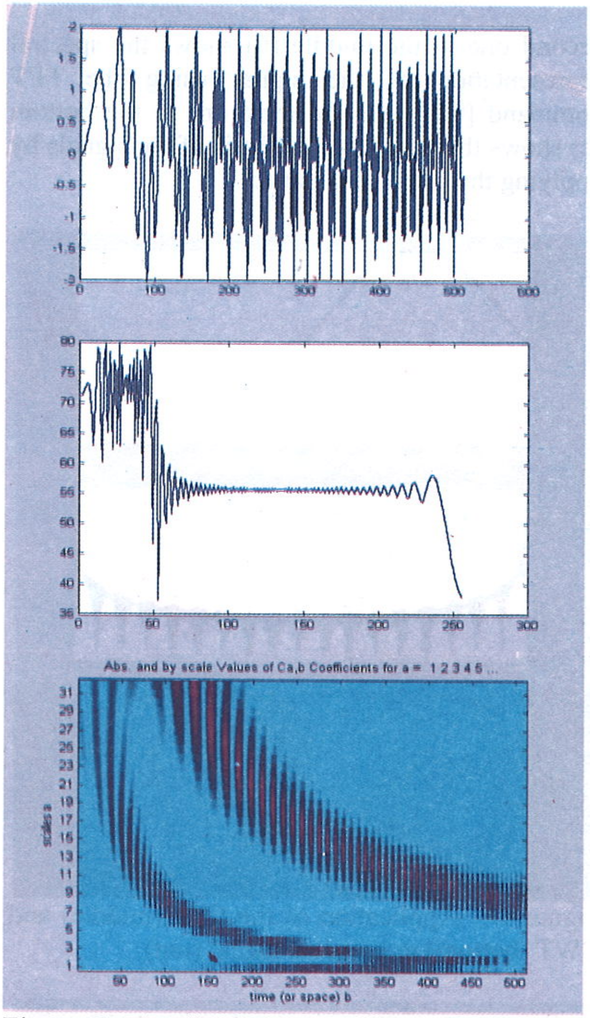

Figure 6 : Application of the CFT (middle) and the CWT (middle) to the chirp signal (top)

For the first case, the CWT picks up obviously the discontinuity at the temporal sample 150 and the sharp transient at the temporal sample 210 while the CFT fails to show these features; this is due to the global time localization of the FT. In the second case the CFT shows no difference between the two spectrums; this can be deduced easily from the spectrum expression of each signals that is : $|\exp(j2\pi f t_0).S(f)| = |S(f)|$; while the CWT detects clearly the shift value since the CWT of a shifted signal is CWT(a, b- t_0).

Let us come back to the third case, the signals are of the form A(t).exp(jΦ(t)) known as the phase modulated signals; a type of signals whose spectra change with time [10, 11]. The first derivative of the phase Φ(t) represents the instantaneous frequency. In this case, the instantaneous frequencies are $\propto_1 t$ and $\propto_2 t$ where the slope $\propto_i$ is called the sweep rate. We have chosen $\propto_1$ and $\propto_2$ with the values of 250 and 50 respectively.
The CWT, by using the Morlet wavelet function, can distinguish obviously the two slopes whereas the CFT shows a meaningless spectrum. This is explained by examining the FT expression of the chirp function, of the general form $\exp(j\pi\propto t^2)$, that is equal to
$(j\propto)^{-1/2}\exp(-j\pi f^2/\propto)$ where the CFT is a superposition of sinusoid functions of frequencies of $f/2\propto_1$ and $f/2\propto_2$ and amplitudes of $(\propto_1)^{-1/2}$ and $(\propto_2)^{-1/2}$ respectively.

6- CONCLUSION

We have presented in this paper an overview of the Wavelet analysis, where we have concentrated our study on the continuous Wavelet transform (CWT) case. The basis upon which we attempt to illustrate the CWT was the Fourier transform. Different criteria have been discussed to distinguish between the two techniques; we have emphasized the time-frequency resolution role. It has been shown that the CWT provides a better performance regarded to the FT for analyzing the non-stationary signals or the time-varying spectra signals and those representing abrupt changes or discontinuities or some shift in time domain, the situations encountered frequently in practice. The main advantage of the CWT is the changed time-scale (time-frequency) information cells sizes depending on the time-frequency localization of the searched information. We have seen that the Wavelet theory is not based upon a specific wavelet function, it deals with any function that satisfies the admissibility and regularity conditions. We have tried to view the CWT from different angles : pure and applied mathematics, signal processing theory, engineering... The wavelet theory is based on a set of contributions and works of different researchers and scientists in several fields, developed mainly since the beginning of the century and exploded since the 80's. the wavelet theory is applied to different fields of the signal processing world. It is really a rich domain of research that seems to be at its youth.

REFERENCES

[1] Daubechies I. "where do wavelet come from? A personal point of view" *proceedings of the IEEE*, **Vol. 4, April 1996**, pp. 510-513

[2] "____ Handbook " CRC Press Inc. 1996. Y. Sheng . Chapter 10. Wavelet Transform pp 748-827.

[3] Jawerth B. and Sweldend W. "An overview on wavelet based muliresolution analysis" *AMS subject Classifications* **42-02, 42C10.**

[4] Mandridake E. "Modélisation par ondelettes orthogonales et utilisation du codage vectoriel pour la compression du signal de parole" Thèse de Doctorat 1992, pp 1-34.

[5] Cohen A. and Kovačevič A. : "Wavelets : The Mathematical Background" *Proceeding of the IEEE*, **Vol. 4 April 1996,** pp. 514-522.

[6] Hess-Nielsen N. and Wickerhauser M. V. "Wavelets and Time-Frequency Analysis" *Proceedings of the IEEE,* **Vol. 4, April 1996,** pp. 523-540.
[7] Benoit-Cattin H. "Contribution à la compression sous bandes : Applications en imagerie médicale 2D et 3D". Thèse de Doctorat 1995. pp. 23-32.
[8] Matalgah M. and Knopp J. "Time-varying spectral analysis of non-stationary signals based on combined wavelet and Fourier transforms" *INT. J. ELECTRONICS,* **Vol. 78 No. 3, 1995,** pp. 463-476.
[9] Missiti M., Missiti Y., Oppenheim G., and Poggi J-M. "Wavelet toolbox : for use with Matlab"
[10] Bourdreaux-Bartels G. "Classified Time-Frequency Representations" _____ 1996
[11] Pola S., Macerata A., Emdin M., and Marchesi C. : "estimation of the power Spectral Density in Non-stationary Cardiovascular Time Series : assessing the Role of the Time-Frequency Representations (TFR)" *IEEE Trans. On Biomed. Eng.,* **Vol. 43, January 1996,** pp.446-459