PURE STATE ENTANGLEMENT ENTROPY IN NONCOMMUTATIVE 2D DE SITTER SPACE TIME

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Abstract

Using the general modified field equation, a general noncommutative Klein-Gordon equation up to the second order of the noncommutativity parameter is derived in the context of noncommutative 2D De Sitter space-time. Using Bogoliubov coefficients and a special technics called conformal time; the boson-antiboson pair creation density is determined. The Von Neumann boson-antiboson pair creation quantum entanglement entropy is presented to compute the entanglement between the modes created presented.

Keywords: ***Noncommutative Klein Gordon equation, Pair creation density, Quantum entanglement, Von Neumann entropy.***

Résumé

En utilisant l'équation modifiée du champ généralisée, une équation générale de Klein-Gordon noncommutative jusqu'au deuxième ordre par rapport au paramètre de la noncommutativité est obtenu dans le contexte de 2D espace-temps de de Sitter. En utilisant les coefficients de Bogoliubov et une technique spéciale nommée time conforme, la densité de la création de paire boson-antiboson a été déterminée. L'entropie de l'entanglement quantique de Von Neumann de création boson-anti-boson a été présentée pour le calcul de l'intrication entre les modes crées.

Mots clés : ***équation de Klein-Gordon noncommutative, densité de création de paires, intrication quantique, entropie de Von Neumann.***

ملخص

باستعمال معادلة معدلة لمعادلة الحقول ، معادلة عامة غير تبديلية لكلين- ڤوردن من الدرجة الثانية بالنسبة للوسيط الغير التبادلي قد أشتقت في اطار 2D و مكان دوسيتر اللاتبادلي . باستعمال معاملات بوڤوليوبوف تقنية خاصة تدعى بالكونفورمال زمن- كثافة خلق الزوج بوزون – ضد بوزون ، فان الأنتروبي لفان نيومان قد أستعملت.

الكلمات المفتاحية : معادلة لكلين- ڤوردن الغير تبديلية ، كثافة خلق الأزواج ، الرابط الكوانتي ، أنتروبي فان نيومان.

I. INTRODUCTION

During the last few years, noncommutative (N.C.) Seiberg–Witten (S.W.) space–time geometry has played an important role in understanding various phenomena for example in particle physics and cosmology [1]–[3]. A great effort has been made in understanding quantum processes in strong fields, where the associated vacuum instability leads to an additional source of quantum processes and could enhance the particle creation. Furthermore, quantum entanglement (Q.E.) has been extensively studied in nonrelativistic flat-space setups and expanding universes [4]–[16]. Increasing interest to the emerging field of relativistic quantum information and entanglement has attracted many people [17]–[23]. Refs. [18] and [19], show that Q.E. of fermionic and bosonic particles in a certain type of Freedman–Robertson– Walker (F.R.W.) universe has been shown to have special k-modes frequencies and antifermions pair creation modes mass dependence. In fact, as it was pointed out in [18], the response of Q.E. to the dynamics of the expansion of the universe is affected by the particular choice of quantum field theory employed and the geometric structure of space–time. Information about the rate and volume of the expansion are codified in the frequency and amount of the entangled modes. To quantify the entanglement created between bosonic modes in noncommutative 2D De Sitter space-time we choose the Von Neumann entropy, which is related to Shannon's measure of information, which is important in the context of the information capacity. It is recommended to mention that a useful interpretation of the Von Neumann entropy is that it represents the minimum number of bits required to store the result of a random variable. The goal of this paper is to study the Von Neumann quantum entanglement boson-antiboson modes created by the dynamics of the noncommutative 2D De Sitter space-time.

II. MATHEMATICAL FORMALISM

The noncommutative space-time is characterized by the operators $\hat{X}$ that satisfy the following noncommutation relation:

$$\left[\hat{x}^{\mu},\hat{x}^{\nu}\right]=i\,\theta^{\mu\nu} \tag{1}$$

where $\theta^{\mu\nu}$ are antisymmetric matrix elements that control the noncommutativity of the space–time. The Klein-Gordon equation is given by:

$$\left[g^{\mu\nu}\partial_{\mu}\partial_{\nu}-m^{2}\right]\psi(x)=0 \tag{2}$$

The approach that we follow in this paper in order to derive the noncommutative Klein-Gordon equation, is based on deforming the scalar density that given by:

$$L=\hat{e}*\left(\hat{g}^{\mu\nu}*\left(\hat{D}_{\mu}\hat{\phi}\right)^{\dagger}*\hat{D}_{\nu}\hat{\phi}+m^{2}\hat{\phi}^{\dagger}*\hat{\phi}\right) \tag{3}$$

By applying the modified field equation, which takes the following form:

$$\frac{\partial L}{\partial\hat{\phi}}-\partial_{\mu}\frac{\partial L}{\partial\left(\partial_{\mu}\hat{\phi}\right)}+\partial_{\mu}\partial_{\nu}\frac{\partial L}{\partial\left(\partial_{\mu}\partial_{\nu}\hat{\phi}\right)}-\partial_{\mu}\partial_{\nu}\partial_{\sigma}\frac{\partial L}{\partial\left(\partial_{\mu}\partial_{\nu}\partial_{\sigma}\hat{\phi}\right)}+\mathrm{O}(\theta^{3})=0 \tag{4}$$

By using the generic field $\hat{\varphi}$, such that:

$$\delta_{\hat{\lambda}}\hat{\phi}=i\hat{\lambda}*\hat{\phi} \tag{5}$$

So, we can get directly the noncommutative Klein-Gordon equation that takes the following form:

$$\left(-g^{\mu\nu}\partial_{\mu}\partial_{\nu}-\frac{1}{\sqrt{-g}}\partial_{\mu}\left(\sqrt{-g}\,g^{\mu\nu}\right)\partial_{\nu}+m^{2}\right)\hat{\phi}$$

$$+\frac{1}{8\sqrt{-g}}\theta^{\alpha\beta}\theta^{\rho\sigma}\partial_{\mu}\left(\partial_{\alpha}\partial_{\rho}\left(\sqrt{-g}\,g^{\mu\nu}\right)\partial_{\beta}\partial_{\sigma}\partial_{\nu}\hat{\phi}\right)$$

$$+\frac{i}{2\sqrt{-g}}\theta^{\alpha\beta}\partial_{\mu}\left(\partial_{\alpha}\left(\sqrt{-g}\,g^{\mu\nu}\right)\partial_{\beta}\partial_{\nu}\hat{\phi}\right)+$$

$$m^{2}\frac{i}{2\sqrt{-g}}\theta^{\alpha\beta}\left(\begin{array}{l}\partial_{\alpha}\sqrt{-g}\,\partial_{\beta}\hat{\phi}\\+\frac{i}{4}\theta^{\rho\sigma}\partial_{\alpha}\partial_{\rho}\sqrt{-g}\,\partial_{\beta}\partial_{\sigma}\hat{\phi}\end{array}\right)=0$$

To define the particle states we should follow the quasi-classical approach of [5] to identify the positive and negative modes frequencies and look for the asymptotic behavior of the solutions at t → 0 and t → ∞. Secondly, we solve the N.C. Klein-Gordon equation and compare the solutions with the above quasi-classical limit. An interesting scenario for discussing the particle creation process, is the noncommutative 2D De Sitter space-time, the metric is given by:

$$dS^{2}=dt^{2}-e^{2Ht}dx^{2} \tag{7}$$

We choose to work in the following parameterization choice of $\theta^{\mu\nu}$ such that:

$$\theta^{\mu\nu}=\begin{pmatrix}0&\theta\\-\theta&0\end{pmatrix} \tag{8}$$

So, it is clear that from this choice we have:

$$\theta^{12}=-\theta^{12}=\theta \tag{9}$$

By replacing in the precedent equation (Eq. (6)) we can get directly the following equation:

$$\partial_t^2 + e^{-2Ht}\partial_x^2 - \frac{1}{e^{2Ht}}\left(2H\,e^{2Ht}\partial_t + m^2\right) + \frac{1}{e^{2Ht}}\theta^2\left(H^3 e^{2Ht}\partial_x^2\partial_t\hat{\phi}\right) + \frac{i}{e^{2Ht}}\theta\left(2H^2 e^{2Ht}\partial_x\partial_t\hat{\phi}\right) + m^2\left(\frac{i}{2e^{2Ht}}\theta\left(2H\,e^{2Ht}\partial_x\hat{\phi} + i\theta H^2 e^{2Ht}\partial_x^2\partial_t\hat{\phi}\right)\right) = 0 \tag{10}$$

In order to the N.C. Klein-Gordon equation we suppose that:

$$\hat{\phi}(t,x) = F(t)e^{i(k_x.x)} \tag{11}$$

which allow us to get the following equation:

$$(-\partial_t^2 - \left(2H + \theta^2 H^3 k_x^2 + 2\theta H^2 k_x\right)\partial_t - e^{-2Ht}k_x^2 - m^2\theta H k_x + \frac{\theta^2}{2}H^2 k_x^2)F(t) = 0 \tag{12}$$

To simplify this equation, we can just collect the parameter, in such way the N.C. Klein-Gordon equation written as the following form:

$$\left(\partial_t^2 + A\partial_t + B + C\,e^{-2Ht}\right)F(t) = 0 \tag{13}$$

with:

$$\begin{aligned} A &= -\left(2H + \theta^2 H^3 k_x^2 + 2\theta H^2 k_x\right) \\ B &= -m^2\theta H k_x + \frac{\theta^2}{2}H^2 k_x^2 \\ C &= -k_x^2 \end{aligned} \tag{14}$$

Working in the conformal time η as the following:

$$\eta = -\frac{1}{H\,e^{Ht}} \tag{15}$$

The equation (Eq. (13)) can be written as the following expression:

$$\left(\frac{\partial^2}{\partial\eta^2} + \frac{\widetilde{A}}{\eta}\frac{\partial}{\partial\eta} + \frac{\widetilde{B}}{\eta^2} + C\right)F(\eta) = 0 \tag{16}$$

Of course with a modification in the parameters “ A ”, “ B ” and “ C ” because we deal with conformal time. The complete solution of the precedent equation (Eq. (16)) is a linear combination of the two solutions, which are just the special functions “Bessel J” and “Bessel Y” as the expression:

$$F(\eta) = C_1\eta^{\frac{1-\widetilde{A}}{2}} J\left(\frac{1}{2}\sqrt{\left(\widetilde{A}-1\right)^2 - 4\widetilde{B}}, \sqrt{C}\eta\right) + C_2\eta^{\frac{1-\widetilde{A}}{2}} Y\left(\frac{1}{2}\sqrt{\left(\widetilde{A}-1\right)^2 - 4\widetilde{B}}, \sqrt{C}\eta\right) \tag{17}$$

To identify the positive and negative frequency modes we use the Bogoliubov transformation, which allow us to write the positive or negative frequency modes in the “in” region as a linear combination of positive and negative modes in the “out” region as the following:

$$\psi_0^+ = \alpha\psi_\infty^+ + \beta\psi_\infty^- \tag{18}$$

with the normalization condition:

$$|\alpha|^2 + |\beta|^2 = 1 \tag{19}$$

Such that the particle creation density $\hat{n}$ expressed in terms of the Bogoliubov coefficients as the following:

$$\hat{n} = \frac{|\beta|^2}{|\alpha|^2} \tag{20}$$

By using a special relation between the Bessel functions, which allow us to define the Bogoliubov coefficients in order to compute the creation density, which is:

$$Y(n,x) = \frac{J(n,x)\cos(n\pi) - J(-n,x)}{\sin(n\pi)} \tag{21}$$

with the parameters:

$$\begin{aligned} n &= \frac{1}{2}\sqrt{\left(\widetilde{A}-1\right)^2 - 4\widetilde{B}} \\ x &= \sqrt{C}\eta \end{aligned} \tag{22}$$

Using this property we can make the correspondence with Bogoliubov transformation, we can define the coefficients α and β, in such way the pair creation density is written as:

$$\hat{n} = \frac{\left|\frac{1}{sin(n\pi)}\right|^2}{\left|\frac{cos(n\pi)}{sin(n\pi)}\right|^2} = \left|\frac{1}{cos(n\pi)}\right|^2 \qquad (23)$$

By replacing the parameter n with its equivalent we find the following expression:

$$\hat{n} = \frac{1}{\left(cos\frac{\pi}{2}\sqrt{(A-1)^2 - 4B}\right)^2} \qquad (24)$$

Over the last decade, the discipline of relativistic quantum information has received much attention; its aim is the study of the resource and tasks of quantum information Science in the context of relativity. In particular finding ways to store and manipulate information is a main goal. The importance of the entanglement comes from its dominance role in a lot of tasks of quantum information, the most important example is the teleportation or use it to fight against the major problem in quantum information called the decoherence. So to quantify the entanglement we have a lot of quantities, which allow us to do it, among them we have the Entanglement entropy, which is defined by the following expression:

$$S_B = log_2\left(\frac{|\gamma_B|\frac{2|\gamma_B|^2}{|\gamma_B|^2-1}}{1-|\gamma_B|^2}\right) \qquad (20)$$

Such that:

$$|\gamma_B|^2 = \hat{n} \qquad (21)$$

III. CONCLUSION

In this paper we have derived the noncommutative Klein-Gordon equation from the modified field equation in the context of noncommutative De Sitter space-time. As an application within the quantum field theory and using the Bogoliubov transformations we have derived the boson-antiboson pair creation density. Using this density we have calculated the Von Neumann entropy that is considered as the best tool to quantify the bipartite quantum entanglement between modes. More studies about relativistic entanglement will be presented in a future paper.

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