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IDENTIFICATION OF AN UNKNOWN PART OF THE BOUNDARY OF AN NAVIER-STOKES SYSTEM BY PUNCTUAL SENTINEL

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Abstract

In this work one interests a method for the identification of a part of the boundary in a parabolic equation (Navier-Stokes equations). By the means of the controllability of an adjoint system one has to identify this part while basing on an observation made on part of the boundary known.

Keywords: *Evaluative system, Weakly Controllability, Observability, Operator, Sentinel Punctual.*

Résumé :

Dans ce travail on intéresse à identifier une partie de la frontière d'un domaine sur lequel on a défini une équation de Navier-Stokes. Par le biais de la contrôlabilité d'un système adjoint on a pour identifier cette partie en se basant sur une observation faite sur une partie de la frontière supposée connue.

Mots clés: *Système évolutif, Contrôlabilité faible, Observabilité, Opérateur, Sentinelle ponctuelle*.

ملخص

نعتبر ميدان جزء من حافته مجهول، و منه الطريقة المتبعة من أجل التعرف على هذا الجزء المجهول من الحافة مؤسسة و مبنية على مفهوم جديد يعرف بطريقة الحارس النقطي و ذلك اعتمادا على نظرية المراقبة الجزئية . كما أن استخدام طريقة النقطة الثابتة كطريقة مساندة للطريقة الأولى يؤدي إلى الحل المرغوب. و في الأخير نتيجة التقارب المحلي لمسالة النقطة الثابتة تعرف الجزء المجهول على الحافة.

الكلمات المفتاحيـــة : نظام تغيري، مراقبة ضعيفة، مشاهدة، مؤثر، حارس نقطي.

1. Construction of the sentinel punctual (Definition, existence and uniqueness of the sentinel)

Let $N \in \{2;3\}$, Ω is a bounded open in $\mathbb{R}^n$ with smooth boundary $\partial\Omega = \Gamma_1 \bigcup D_0$, with $\Gamma_1 \bigcap D_0 = \varnothing$ such that D_0 unknown part. Let $O = \{b\} \subset \Omega$, considered as an observatory. $T > 0$ is fixed, we then denote by

$$Q = \Omega \times]0,T[;\Sigma_1 = \Gamma_1 \times]0,T[;\Sigma_0 = D_0 \times]0,T[$$

It is well known that the following Navier-Stokes system

$$y' - \Delta y + y\nabla y + \nabla p = 0 \text{ in } Q \quad (1)$$

And, div $y = 0$ in Q; $y = g$ on Σ_1; $y = 0$ on Σ_0; $y(0) = 0$ in Ω

Where y is flow velocity, and p is the pressure.

We define Ω_τ open ''neighbor'' of Ω of boundary $\partial\Omega_\tau = (\Gamma - D_0) \bigcup D_\tau$.

Where D_τ is defined from D_0 like the place of the points

$$D_\tau = \{b + \tau\alpha(b)\nu(b), b \in D_0\} \quad (2)$$

We denote by ν the outer normal on Γ, τ small real parameter, and α is a C^1 on D_0 with $|\alpha(b)| \le 1, \alpha = 0$ on ∂D_0.

We then denote by

$$Q_\tau = \Omega_\tau \times]0,T[;\Sigma_1 = (\Gamma - D_0) \times]0,T[; \quad \Sigma_\tau = D_\tau \times]0,T[\quad (3)$$

Let $y = y(b,t;\tau)$ be the solution of

$$y' - \Delta y + y\nabla y + \nabla p = 0 \text{ in } Q_\tau \quad (4)$$

And, div $y = 0$ in Q_τ; $y = g$ on Σ_1; $y = 0$ on Σ_τ; $y(0) = 0$ in Ω_τ

We suppose that (4) has a unique solution denoted by $y(\tau) := y(b,t;\tau)$ is some relevant space. The question is Lions which is an other attempt and brings better answer to question **(q)**, as we will explain now:

For a control function in $u \in U = L^2(]0,T[)$, we define the functional

$$S(\lambda,\tau) = \int_0^T u(\lambda) y(b,t;\tau)\,dt \quad (5)$$

Then, firstly, the problem consists in looking for u such that the following conditions are satisfied

$$\left|\frac{\partial}{\partial\tau} S(.,.)\right| \le \varepsilon \text{; with } \varepsilon > 0 \text{ sufficiently small parameter} \quad (6)$$

Secondly, u of minimal norm in U, i.e.

$$\|u(\lambda)\|_U = \text{minimum} \quad (7)$$

Let S be the real function defined by (5). S is said to be a punctual sentinel if there exists $u \in U$ such that properties (6)-(7) are valid.

2. Equivalent controllability problem

The function $y_\tau = \dfrac{\partial y}{\partial\tau}$ solves the problem

$$\frac{\partial}{\partial t} y_\tau - \Delta y_\tau + \nabla(y_\tau \otimes y + y \otimes y_\tau) + \nabla p_\tau = 0 \text{ in } Q_\lambda \quad (8)$$

And, div $y_\tau = 0$ in Q_λ; $y_\tau = 0$ on Σ_1; $y_\tau = -\dfrac{\partial}{\partial\nu} y(\lambda)$ on Σ_λ; $y_\tau(0) = 0$ in Ω_λ

$y(\lambda) = y(b,t;\lambda)$ solves (4). Thus $\dfrac{\partial}{\partial\tau} S(\lambda,\lambda)$ data by:

(q) How to identification of an unknown boundary (D_0)?

We now consider the sentinel method of

We set $Dq = \nabla q + \nabla q^t$ and introduce the adjoint state system associated to (8)

$$-\frac{\partial}{\partial t} q - \Delta q - Dqy(\lambda) + \nabla\pi = u\delta_b \text{ in } Q_\lambda \quad (10)$$

And, div $q = 0$ in Q_λ; $q = 0$ on Σ_1; $q = 0$ on Σ_λ; $q(T) = 0$ in Ω_λ

Therefore, let q be the unique solution, it is well known that

$$q \in L^2\left(0,T;\left(H_0^1(\Omega_\lambda)\right)^N\right) \bigcap C^0\left(0,T;\left(L^2(\Omega_\lambda)\right)^N\right)$$

Depends on u which is to be determined.

Indeed, if we multiply the first equation in (10) by y_τ, and we integrate by parts over $(0,T)$, we obtain

$$\int_0^T u(\lambda) y_\tau(b,t)\,dt = -\int_{\Sigma_\lambda} \frac{\partial q}{\partial\nu_*} y_\tau \,d\Sigma$$

Indeed from (8) we have,

$$\int_0^T u(\lambda) y_\tau(b,t)\,dt = -\int_{\Sigma_\lambda} \frac{\partial q}{\partial\nu_*} y_\tau \,d\Sigma = \quad (11)$$

$$= \int_{\Sigma_\lambda} \left(\alpha \frac{\partial}{\partial\nu} y(\lambda)\right) \frac{\partial q}{\partial\nu_*} d\Sigma$$

Finally, if we define the linear continuous operator :

$$\frac{\partial}{\partial\tau} S(\lambda,\lambda) = \int_0^T u(\lambda) y_\tau(b,t)\,dt \quad (9)$$

$$= \int_{\Sigma_\lambda} \left(\alpha \frac{\partial}{\partial \nu} y(\lambda) \right) \frac{\partial q}{\partial \nu_*} d\Sigma$$

$$= Bu(\lambda)$$

One has: $\frac{\partial}{\partial \tau} S(\lambda, \lambda) = Bu(\lambda)$

This is a control problem?

Proof of: B^* ''adjoint of B '' is injective.

i.e. $\ker(B^*) = \{0\}$.

Operator B define by

$$Bu = \int_{\Sigma_\lambda} \left(\alpha \frac{\partial}{\partial \nu} y(\lambda) \right) \frac{\partial q}{\partial \nu_*} d\Sigma$$

Whose adjoint is

$$B^* \sigma = \chi_{(0,T)} z$$

And where $\chi_{(0,T)}$ denotes the characteristic function of $(0,T)$,

$$(\chi_{(0,T)} z)(t) = \begin{cases} z(t) & if \quad t \in (0,T) \\ 0 & otherwise \end{cases}$$

And z be the solution of

$$\frac{\partial}{\partial t} z - \Delta z + z \nabla z + \nabla p = 0 \text{ in } Q_\lambda \qquad (13)$$

And, $\operatorname{div} z = 0$ in Q_λ; $z = 0$ on Σ_1; $z = -\left(\alpha \frac{\partial}{\partial \nu} y(\lambda) \right) \sigma$ on Σ_λ; $z(0) = 0$ in Ω_λ

So that from (12) and (13) we deduce :

$$B : U \to E = \mathbb{R}$$

$$u \to Bu = \int_{\Sigma_\lambda} \left(\alpha \frac{\partial}{\partial \nu} y(\lambda) \right) \frac{\partial q}{\partial \nu_*} d\Sigma \qquad (12)$$

The equation (11) allows to rewrite (9) into

$$\frac{\partial}{\partial \tau} S(\lambda, \lambda) = \int_0^T u(\lambda) y_\tau(b,t)\, dt =$$

$$\chi_{(0,T)} z = B^* \sigma$$

Suppose now that $\chi_{(0,T)} z = B^* \sigma = 0$ i.e.

$z = 0$ in $(0,T)$.

$z = 0$ in $(0,T)$. Thus $\left(\alpha \frac{\partial}{\partial \nu} y(\lambda) \right) \sigma = 0$

$\left(\alpha \frac{\partial}{\partial \nu} y(\lambda) \right) \sigma = 0 \Leftrightarrow \alpha = 0$ or $\frac{\partial}{\partial \nu} y(\lambda) = 0$ or $\sigma = 0$.

This equality must take place for any regular function α, with $|\alpha(b)| \le 1$, $\alpha = 0$ on ∂D_0. That is equivalent to

$\frac{\partial}{\partial \nu} y(\lambda) = 0$ or $\sigma = 0$.

$\frac{\partial}{\partial \nu} y(\lambda) \neq 0$ otherwise $y(\lambda) = 0$ in Q_λ.

Then we have: $\sigma = 0$.

Then we deduce B^* is injective.

$\ker(B^*) = \{0\}$ is equivalent to $\overline{\operatorname{Im}(B)} = \mathbb{R}$ i.e.

$$\forall \varepsilon > 0, \forall x \in \mathbb{R}, \exists u \in U; |Bu - x| \le \varepsilon \qquad (14)$$

$$(u,z)_U = \sigma \int_{\Sigma_\lambda} \left(\alpha \frac{\partial}{\partial \nu} y(\lambda) \right) \frac{\partial q}{\partial \nu} d\Sigma = (\sigma, Bu)_E$$

We then obtain

closed in U from (14).

and let $u(\tau)$ be the solution of the following minimization problem

$$\min \frac{1}{2} \|u\|_U^2 ; u \in u_{ad} \qquad (15)$$

Let F and G be two functions defined as

$F(u) = \frac{1}{2} \|u\|_U^2$ and

$$G(\mu) = \begin{cases} 0 & if \quad |\mu - x| \le \varepsilon \\ +\infty & otherwise \end{cases}$$

So that from (15) we deduce

$$\min F(u) + G(Bu); \ u \in U$$

Applying the duality of Fenchel and Rockafeller [7], one gets

$$u(\tau) = B^* \sigma^* \qquad (16)$$

and let σ^* be the solution of the dual minimization problem

$$\min F^*(B^* \sigma) + G^*(-\sigma); \ \sigma \in E \qquad (17)$$

With F^* and G^* being the Fenchel conjugates of F and G.

Such that $F^* = F$ and G^* defined by

$$G^*(\sigma) = (x, \sigma)_E + \varepsilon \sigma$$

Then of course (15) becomes :

By a convex duality process a control fulfilling the conditions (6)-(7) is exhibited.

It remains to construct $u(\tau)$ as the function of minimal norm satisfying (14).

Let

$u_{ad} = \{u \in U$ such that $|Bu - x| \leq \varepsilon, x \in \mathbb{R}\}$ Then u is a nonempty set. And convex and For any $\delta\sigma \in \mathbb{R}$ and $\sigma \neq 0$, one has

$$\left(\frac{\partial J}{\partial \sigma}, \delta\sigma\right)_E = \left(BB^*\sigma + \varepsilon - x, \delta\sigma\right)_E$$

For $\sigma = \sigma^*$ one has $BB^*\sigma^* + \varepsilon - x$ We have

$$BB^*\sigma^* - x = -\varepsilon \qquad (19)$$

Since $u(\lambda) = B^*\sigma^*$, we have $Bu(\lambda) - x = -\varepsilon$

Thus $|Bu(\lambda) - x| = \varepsilon$

We will have that $|x| > \varepsilon$. Eventually (19) gives

$$BB^*\sigma^* = x - \varepsilon \Leftrightarrow Bu(\lambda) = x - \varepsilon$$

$$BB^*\sigma^* = x - \varepsilon \Leftrightarrow \frac{\partial}{\partial \tau} S(\lambda, \lambda) = x - \varepsilon$$

Choosing x, such that $\left|\frac{\partial}{\partial \tau} S(\lambda, \lambda)\right| \leq \varepsilon$

3. A use of the concept of sentinel: the identification of the unknown boundary :

$$\min J(\sigma) = F(z) + \varepsilon\sigma - (x, \sigma)_E ;\ \sigma \in E \qquad (18)$$

Where z solves the problem (13).

For $\sigma^* \neq 0$ then it is supposed that $|x| > \varepsilon$. (Otherwise, $\sigma^* = 0$ solves of (18) $\Leftrightarrow |x| \leq \varepsilon$).

$$S(\lambda, \tau) = S(\lambda, \lambda) + (\tau - \lambda)\frac{\partial}{\partial \tau} S(\lambda, \lambda) + \circ(|\tau - \lambda|)$$

The observation is y in point b, for the time T. we denote by y_{obs} this observation, and for the sake of simplicity, we suppose the existence of τ_s such that

$$y_{obs} = y(b, t; \tau_s) = m_0 \in L^2(]0, T[) \qquad (20)$$

Let S_{obs} be the global information provided by the observation y_{obs}.

$$S_{obs}(\lambda, \tau_s) = \int_0^T u(\lambda) m_0 \, dt$$

In particular for $\tau = \tau_s$ and so that (6) becomes

$$S_{obs}(\lambda, \tau_s) = S(\lambda, \lambda) + (\tau_s - \lambda)\frac{\partial}{\partial \tau} S(\lambda, \lambda) + \circ(|\tau_s - \lambda|)$$

One gets

$$S_{obs}(\lambda, \tau_s) \simeq S(\lambda, \lambda) + (\tau_s - \lambda)\varepsilon + \circ(|\tau_s - \lambda|)$$

Choosing $E = \ell^2(\mathbb{R})$, and

$$g(\lambda_k) = \lambda_{k+1} = S_{obs}(\lambda_k, \tau_s) - S(\lambda_k, \lambda_k) \qquad (21)$$

Where g is a mapping from E to itself obviously defined from (21) and (5).

Let us now, present a use of the concept of sentinel applied for give approximation of the shape of D_0.

Let $S(\lambda, \tau)$ the punctual sentinel in the sense of J.L.Lions [2]. Indeed

$$S(\lambda, \tau) \simeq S(\lambda, 0) + \tau \frac{\partial}{\partial \tau} S(\lambda, 0)$$

Differentiating $S(\lambda, \tau)$ with respect to τ at the point (λ, λ) one gets

$$g'(\lambda) = \frac{\partial}{\partial \tau_s} S(\lambda, \tau_s) - \frac{\partial}{\partial \tau_s} S(\lambda, \lambda) - \frac{\partial}{\partial \tau} S(\lambda, \lambda)$$

Thus

$$g'(\tau_s) = \frac{\partial}{\partial \tau_s} S(\tau_s, \tau_s) - \frac{\partial}{\partial \tau_s} S(\tau_s, \tau_s) - \frac{\partial}{\partial \tau} S(\tau_s, \tau_s) = -\frac{\partial}{\partial \tau} S(\tau_s, \tau_s)$$

So that from (6) we deduce $|g'(\tau_s)| \leq \varepsilon < 1$

Then the sequence (λ_k) locally converging to τ_s. This will give an approximation of the shape of D_{τ_s}.

we deduce the value of τ_s and thus one chooses $D_0 = D_{\tau_s}$.

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