

ON FOURIER-PARSEVAL KERNELS PART I.

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Résumé

Dans ce travail, nous précisons en premier lieu la méthode des noyaux de Fourier – Parseval. Nous appliquons ensuite cette technique pour étudier une classe de noyaux pour lesquels les intégrales de Poisson, Wallis et Euler sont des cas particuliers.

Des propriétés et des identités remarquables sont ainsi établies.

Mots clés: *Séries de Fourier, coefficients de Fourier, Fonctions analytiques, Fonctions génératrices, Séries et Intégrales.*

Abstract

First of all, we point out in this paper, the Fourier-Parseval Kernels method. Then we apply this technique in order to study a class of kernels for which the Poisson Wallis and Euler's integrals are particular cases.

Thus, remarkable properties and identities are established.

Keywords: *Fourier series, Fourier coefficients, Analytic and Generating functions, Series and Integrals.*

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In this paper, we use the technique called Fourier - Parseval kernels, to deduce some remarkable integrals, series, harmonic functions and Fourier expansions. For instance, we show that Euler, Wallis and Poisson's integrals are particular cases of kernels. Generally there exists a whole class of such kernels, we may find other results and Tables in [1].
Let

$$f(z) = \sum_{n \geq 0} a_n z^n$$

be an analytic function with its Taylorian expansion around the origin and D_c its convergence domain.

For

$$z = re^{i\theta} \Rightarrow f(re^{i\theta}) = \sum_{n \geq 0} a_n r^n (\cos n\theta + i \sin n\theta) \quad (1)$$

For symmetry reasons, we can choose $\theta \in [-\pi, \pi]$ instead of $[0, 2\pi]$

Then:

$$\operatorname{Re} f(re^{i\theta}) = \sum_{n \geq 0} a_n r^n \cos n\theta \quad (2)$$

and

$$\operatorname{Im} f(re^{i\theta}) = \sum_{n \geq 1} a_n r^n \sin n\theta \quad (3)$$

The series (1), (2) and (3) are of Fourier type.

If the convergence of (1), (2) and (3) is uniform within D_c , then we know that (2) and (3) are exactly the Fourier expansion of $\operatorname{Re} f(re^{i\theta})$ and

$\operatorname{Im} f(re^{i\theta})$. Otherwise, we apply the uniqueness theorem of Riemann on Fourier developments [8] and [9] to deduce, again, under some hypothesis of integrability that in both cases (2) and (3) are the Fourier expansions of $\operatorname{Re} f(re^{i\theta})$ and $\operatorname{Im} f(re^{i\theta})$.

Then we write write,

$$\operatorname{Re} f(re^{i\theta}) = \sum_{n \geq 0} a_n r^n \cos n\theta ; \theta \in [-\pi, \pi] \quad (4)$$

$$\operatorname{Im} f(re^{i\theta}) = \sum_{n \geq 1} a_n r^n \sin n\theta ; \theta \in [-\pi, \pi] \quad (5)$$

The harmonic functions $\operatorname{Re} f(re^{i\theta})$ and $\operatorname{Im} f(re^{i\theta})$ are even and odd respectively. Both Fourier coefficients are identical, excepting a_0 .

ملخص

في البداية نعرض تقنية نواة فورييه-بارسفال- (Parseval) -Fourier بعد ذلك نطبق هذه الطريقة لدراسة فئة من هذه النواة حيث تكاملات أولار (Euler) واليس (Wallis) و بواسون (Poisson) تعتبر حالات خاصة منها. ثم نثبت و نبرهن بعض الخواص و التطابقات.

الكلمات المفتاحية: سلاسل فورييه، معاملات فورييه، التوابع التحليلية، التوابع المولدة، السلاسل و تكاملات

Some Fourier developments are obtained from analytic continuation as,

$$f(iz) = \sum_{n \geq 0} a_n (iz)^n$$

this gives an expansion which is neither even nor odd.

Therefore, from the series (4) and (5) we deduce the Fourier coefficients formulas:

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Re} f(re^{i\theta}) d\theta ; \quad \forall r \in D_c \quad (6) \\ &\frac{1}{\pi} \int_{-\pi}^{\pi} \operatorname{Re} f(re^{i\theta}) \cos n\theta d\theta \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \operatorname{Im} f(re^{i\theta}) \sin n\theta d\theta \\ &= a_n r^n \quad \forall r \in D_c ; \quad \forall n \in \mathbb{N}^* \quad (7) \end{aligned}$$

Finally, Parseval identity gives :

$$\int_{-\pi}^{\pi} [\operatorname{Re} f(re^{i\theta})]^2 d\theta = \pi \left(2a_0^2 + \sum_{n \geq 1} a_n^2 r^{2n} \right) \quad (8)$$

and

$$\int_{-\pi}^{\pi} [\operatorname{Im} f(re^{i\theta})]^2 d\theta = \pi \sum_{n \geq 1} a_n^2 r^{2n} ; \quad \forall r \in D_c \quad (9)$$

The simplest application is the geometric series

$$\left(\frac{1}{1-z} \right) = \sum_{n \geq 0} z^n ; \quad |z| < 1$$

we get

$$\begin{aligned} \frac{1-r \cos \theta}{1+r^2-2r \cos \theta} &= \sum_{n \geq 0} r^n \cos n\theta , \\ \frac{r \sin \theta}{1+r^2-2r \cos \theta} &= \sum_{n \geq 1} r^n \sin n\theta \end{aligned}$$

Such that ; $|\theta| \leq \pi$ and $r \in [0, 1[$.

We also have ,

$$\int_{-\pi}^{\pi} \frac{1-r \cos \theta}{1+r^2-2r \cos \theta} d\theta = 2\pi ; \quad \forall r \in [0, 1[,$$

$$\begin{aligned} \int_{-\pi}^{\pi} \left(\frac{1-r \cos \theta}{1+r^2-2r \cos \theta} \right) \cos n\theta d\theta &= \int_{-\pi}^{\pi} \left(\frac{r \sin \theta}{1+r^2-2r \cos \theta} \right) \sin n\theta d\theta = \pi r^n \\ \forall n \in \mathbb{N}^* , \forall r \in [0, 1[\end{aligned}$$

Parseval equality is :

$$\begin{aligned} \int_{-\pi}^{\pi} \left[\frac{1-r \cos \theta}{1+r^2-2r \cos \theta} \right]^2 d\theta &= \pi \left(2 + \frac{r^2}{1-r^2} \right) ; \quad \forall r \in [0, 1[\\ \int_{-\pi}^{\pi} \left[\frac{r \sin \theta}{1+r^2-2r \cos \theta} \right]^2 d\theta &= \pi \left(\frac{r^2}{1-r^2} \right) ; \quad \forall r \in [0, 1[\end{aligned}$$

Even and odd properties imply integrals from 0 to π .

Secondly, the choice of $\frac{1}{1+z}$; implies the results for all

$$|r| < 1 .$$

For instance, the Poisson integral

$$\int_0^{\pi} \lg \left(1+r^2-2r \cos \theta \right) d\theta = 0 ; \quad (|r| < 1)$$

is just the Fourier kernel of the analytic development.

$$\lg(1-z) = - \sum_{n \geq 1} \frac{z^n}{n} \quad \text{with } |z| < 1$$

Wallis integrals

$$\int_0^{\frac{\pi}{2}} \cos^n \theta d\theta = \int_0^{\frac{\pi}{2}} \sin^n \theta d\theta$$

are particular cases of kernels associated to the Binome formulae

$$(1+z)^\alpha = \sum_{n \geq 0} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} z^n ; \quad (|z| < 1)$$

Euler's integrals are particular cases of Poisson's integrals, obtained when the parameter r tends to 1.

Moreover, Parseval identity give s us the computation :

$$\int_0^{\pi} \frac{1}{4} \lg^2(1+r^2-2r \cos \theta) d\theta = \int_0^{\pi} \left[\arctan \frac{r \sin \theta}{1-r \cos \theta} \right]^2 d\theta = \frac{\pi}{2} \sum_{n \geq 1} \frac{r^{2n}}{n^2} \quad (|r| < 1)$$

The rich class of analytic Taylorian expansions of functions, implies an enormous choice of Fourier - Parseval kernels. Other developments and details may be found in [1].

Now, we would like to apply this technique in order to establish some results, as shown above, related to series, integrals, harmonic functions, and Fourier expansions of remarkable analytic functions.

In a forthcoming paper [2], other generalizations and extensions are studied. Commentaries, summary and left open problems are quoted therein.

Proposition 1: We have the relations :

$$\begin{aligned} \text{a) } \int_0^{\pi} (1+r^2-2r \cos \theta)^\alpha d\theta &= \pi \sum_{n \geq 0} \left[\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} \right]^2 r^{2n} ; \\ &\forall \alpha \in \mathbb{R} ; \quad \forall |r| < 1 \\ \text{b) } \int_0^{\frac{\pi}{2}} \cos^\alpha \theta d\theta &= \int_0^{\frac{\pi}{2}} \sin^\alpha \theta d\theta = \frac{\pi}{2^{\alpha+1}} \sum_{n \geq 0} \left[\frac{\alpha(\alpha-2)(\alpha-4)\dots(\alpha-2(n-1))}{2^n n!} \right]^2 ; \\ &\forall \alpha > -1 \end{aligned}$$

Proof. By applying Fourier - Parseval technique (as explained in the Introduction) to the Binome expansion

$$(1+z)^\alpha = \sum_{n \geq 0} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} z^n ; \quad |z| < 1$$

We deduce the following kernels :

$$\begin{aligned} & (1+r^2+2r\cos\theta)^{\frac{\alpha}{2}} \cos(\alpha\omega_{r,\theta}) \\ &= 1 + \sum_{n \geq 1} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} r^n \cos(n\theta) \end{aligned} \quad (10)$$

$$\begin{aligned} & (1+r^2+2r\cos\theta)^{\frac{\alpha}{2}} \sin(\alpha\omega_{r,\theta}) \\ &= \sum_{n \geq 1} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} r^n \sin(n\theta) \end{aligned} \quad (11)$$

where $\omega_{r,\theta} = \arctan\left(\frac{r \sin \theta}{1+r \cos \theta}\right)$

and $\forall |r| < 1 ; \forall \alpha \in \mathbb{R} ; \forall |\theta| \leq \pi$

Fourier coefficients of (10) and (11) are well determined. The two values ($\pm\pi$) of θ are deduced from the Dirichlet continuity conditions.

In particular, from a_0 we have

$$\int_0^\pi (1+r^2+2r\cos\theta)^{\frac{\alpha}{2}} \cos(\alpha\omega_{r,\theta}) d\theta = \pi ; \forall |r| < 1 ; \forall \alpha \in \mathbb{R} \quad (12)$$

Parseval identity implies:

$$\begin{aligned} \int_0^\pi (1+r^2+2r\cos\theta)^{\frac{\alpha}{2}} \cos^2(\alpha\omega_{r,\theta}) d\theta &= \frac{\pi}{2} \left[2 + \sum_{n \geq 1} \left(\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} \right)^2 r^{2n} \right] \\ \int_0^\pi (1+r^2+2r\cos\theta)^{\frac{\alpha}{2}} \sin^2(\alpha\omega_{r,\theta}) d\theta &= \frac{\pi}{2} \sum_{n \geq 1} \left(\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} \right)^2 r^{2n} \\ &\forall |r| < 1 ; \forall \alpha \in \mathbb{R} \end{aligned}$$

Hence, we deduce :

$$\begin{aligned} \int_0^\pi (1+r^2+2r\cos\theta)^\alpha d\theta &= \pi \sum_{n \geq 0} \left(\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} \right)^2 r^{2n} ; \\ &\forall |r| < 1 ; \forall \alpha \in \mathbb{R} ; \end{aligned} \quad (13)$$

this proves (a)

On the external point of the interval $r = 1$, the series in (13) above is convergent for $\alpha > -1/2$ and divergent for $\alpha \leq -1/2$.

Therefore, from Abel and Lebesgue's convergence theorems we deduce as r tends to 1 that:

$$\begin{aligned} \int_0^\pi (1+\cos\theta)^\alpha d\theta &= \frac{\pi}{2^\alpha} \sum_{n \geq 0} \left[\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} \right]^2 ; \forall \alpha > -\frac{1}{2} \\ \int_0^{\frac{\pi}{2}} \cos^{2\alpha} \theta d\theta &= \frac{\pi}{2^{2\alpha+1}} \sum_{n \geq 0} \left[\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} \right]^2 ; \forall \alpha > -\frac{1}{2} \end{aligned}$$

And finally :

$$\int_0^{\frac{\pi}{2}} \cos^n \theta d\theta = \frac{\pi}{2^{\alpha+1}} \sum_{n \geq 0} \left[\frac{\alpha(\alpha-2)(\alpha-4)\dots(\alpha-2(n-1))}{2^n n!} \right]^2 ; \forall \alpha > -1 \quad (14)$$

this establishes (b)

Proposition 1 expresses a Fourier - Parseval kernel formulae, which is at the same time a generating function ([2] and [5]). In other words the integral as a function of the variable little r is analytic and the series in the right handside contains fully determined Taylor coefficients.

Secondly, it is an extension of Wallis integrals from \mathbb{N} to \mathbb{R} .

As a corollary, we have the following result :

Corollary 1 :

$$\text{a) } \sum_{n=0}^p \left[\frac{p(p-1)(p-2)\dots(p-(n-1))}{n!} \right]^2 = \frac{(2p)!}{(p!)^2} ; \forall p \in \mathbb{N}$$

$$\text{b) } \sum_{n \geq 0} \left[\frac{(2p+1)(2p-1)(2p-3)\dots(2p+1-2(n-1))}{2^n n!} \right]^2 = \frac{1}{\pi} \frac{2^{2(2p+1)} (p!)^2}{(2p+1)!} ; \forall p \in \mathbb{N}$$

Proof. Let us remind that Wallis integrals are:

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n \theta d\theta = \int_0^{\frac{\pi}{2}} \sin^n \theta d\theta ; \forall n \in \mathbb{N}$$

Where,

$$I_{2p} = \frac{\pi}{2} \frac{(2p)!}{2^{2p} (p!)^2} \quad \text{and} \quad I_{2p+1} = \frac{2^{2p} (p!)^2}{(2p+1)!} \quad p \in \mathbb{N}$$

Hence the finite sum in (a) is obtained from (b) of Prop 1 by taking $\alpha = 2p$.

In the same manner (b) is derived from (b) of Prop 1, by putting $\alpha = 2p + 1$.

Proposition 2 : We have :

$$\text{(a) } (1+r^2)^\alpha \sum_{n \geq 0} \frac{\alpha(\alpha-1)\dots(\alpha-2n+1)}{(n!)^2} \left(\frac{r}{1+r^2} \right)^{2n} = \sum_{n \geq 0} \left[\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} \right]^2 r^{2n} ; \forall |r| < 1 ; \forall \alpha \in \mathbb{R}$$

$$\text{(b) } \sum_{n \geq 0} \left[\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} \right]^2 = 2^\alpha \sum_{n \geq 0} \frac{\alpha(\alpha-1)\dots(\alpha-2n+1)}{2^{2n} (n!)^2} ; \forall \alpha > -\frac{1}{2}$$

Proof. Let

$$I_{r,\alpha} = \int_0^\pi (1+r^2+2r\cos\theta)^\alpha d\theta ; \forall r \in \mathbb{R} ; \alpha \in \mathbb{R}$$

$$\Rightarrow I_{r,\alpha} = (1+r^2)^\alpha \int_0^\pi \left(1 + \frac{2r}{1+r^2} \cos\theta \right)^\alpha d\theta ;$$

because $\left| \frac{2r}{1+r^2} \right| < 1 ; \forall r \in \mathbb{R} ; |r| \neq 1$; using the binome

development and uniform convergence , we get

$$\begin{aligned} I_{r,\alpha} &= (1+r^2)^\alpha \sum_{n \geq 0} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} \left(\frac{2r}{1+r^2} \right)^n \int_0^\pi \cos^n \theta d\theta. \\ \int_0^\pi \cos^n \theta d\theta &= \int_0^{\frac{\pi}{2}} \cos^n \theta d\theta + \int_{\frac{\pi}{2}}^\pi \cos^n \theta d\theta = (1+(-1)^n) \int_0^{\frac{\pi}{2}} \cos^n \theta d\theta. \end{aligned}$$

Therefore :

$$I_{r,\alpha} = (1+r^2)^\alpha \sum_{n \geq 0} \frac{2^\alpha \alpha(\alpha-1)\dots(\alpha-2n+1)}{(2n!)^2} \left(\frac{2r}{1+r^2} \right)^{2n} \frac{\pi}{2} \frac{(2n)!}{2^{2n} (n!)^2}$$

$$\Rightarrow I_{r,\alpha} = \pi(1+r^2)^\alpha \sum_{n \geq 0} \frac{\alpha(\alpha-1)\dots(\alpha-2n+1)}{(n!)^2} \left(\frac{r}{1+r^2}\right)^{2n}$$

By continuity and uniform convergence, this relation holds for all $r \in \mathbb{R}$

For the values of $r \in \mathbb{R}$ satisfying $|r| < 1$ and using

Proposition 1, we get :

$$\begin{aligned} I_{\alpha,r} &= \pi \sum_{n \geq 0} \left[\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{(n!)} \right]^2 r^{2n} \\ &= \pi(1+r^2)^\alpha \sum_{n \geq 0} \frac{\alpha(\alpha-1)\dots(\alpha-2n+1)}{(n!)^2} \left(\frac{r}{1+r^2}\right)^{2n} ; \forall \alpha \in \mathbb{R} ; |r| < 1 \end{aligned}$$

Using Abel's theorem as r tends to 1, we get (b) and the proof is complete.

Proposition 2 above, shows that for numerical applications the integral,

$$\int_0^\pi (1+r^2+2r \cos \theta)^\alpha d\theta = I_{r,\alpha} ; \alpha \in \mathbb{R}$$

may be computed for all values of $r \in \mathbb{R}$, and α in the corresponding domain of convergence. However, the result is not of functional nature, neither does it express that the integral is a Fourier - Parseval kernel.

On the other hand, for $|r| < 1$, the proposition shows that the function :

$$f_\alpha(r) = (1+r^2)^\alpha \sum_{n \geq 0} \frac{\alpha(\alpha-1)\dots(\alpha-2n+1)}{(n!)^2} \left(\frac{r}{1+r^2}\right)^{2n} ; \alpha \in \mathbb{R}$$

is a generating and analytic function, with respect to the variable r .

Corollary 2:

$$(a) \sum_{n \geq 0} \frac{p(p-1)\dots(p-(2n-1))}{2^{2n}(n!)^2} = \frac{(2p)!}{2^p(p!)^2} ; p \in \mathbb{N}$$

$$(b) \sum_{n \geq 0} \frac{(2p+1)(2p-1)\dots(2p+1-2(2n-1))}{2^{4n}(n!)^2} = \frac{1}{\pi} \frac{2^{\frac{3}{2}(2p+1)}(p!)^2}{(2p+1)!} ; p \in \mathbb{N}$$

Proof. From the relation (b) of Prop 2, a change of variable from α to $\alpha/2$ gives :

$$\sum_{n \geq 0} \left[\frac{\alpha(\alpha-2)\dots(\alpha-2(n-1))}{2^n n!} \right]^2 = 2^{2\alpha} \sum_{n \geq 0} \frac{\alpha(\alpha-2)\dots(\alpha-2(2n-1))}{2^{4n}(n!)^2} ; \forall \alpha > -1$$

Taking $\alpha = 2p$; $p \in \mathbb{N}$, we get

$$\sum_{n \geq 0} \frac{p(p-1)\dots(p-(2n-1))}{2^{2n}(n!)^2} = 2^{-p} \sum_{n \geq 0} \left[\frac{p(p-1)\dots(p-(n-1))}{n!} \right]^2$$

Then Corollary 2 implies

$$\sum_{n \geq 0} \frac{p(p-1)\dots(p-(2n-1))}{2^{2n}(n!)^2} = \frac{(2p)!}{2^p(p!)^2} ; p \in \mathbb{N}$$

This proves (a), which is a finite sum according to p being even or odd.

In the same manner, for (b) we take $\alpha = 2p + 1$. This implies that :

$$\sum_{n \geq 0} \frac{(2p+1)(2p-1)\dots(2p+1-2(2n-1))}{2^{4n}(n!)^2} = 2^{-\frac{3p-1}{2}} \sum_{n \geq 0} \left[\frac{(2p+1)(2p-1)\dots(2p+1-2(n-1))}{2^n n!} \right]^2$$

and corollary 1 implies :

$$\sum_{n \geq 0} \frac{(2p+1)(2p-1)\dots(2p+1-2(2n-1))}{2^{4n}(n!)^2} = \left(\frac{1}{\pi \cdot 2^{\frac{2p-1}{2}}} \right) \left(\frac{2^{2(2p+1)}(p!)^2}{(2p+1)!} \right) ; p \in \mathbb{N}$$

and the proof is complete.

Proposition 3 :

$$(a) \sum_{n \geq 1} \frac{(2n)!}{2^{2n}(n!)^2} \left(\frac{r}{1+r^2}\right)^{2n} = \lg(1+r^2) ; \forall |r| < 1$$

$$(b) \sum_{n \geq 1} \frac{(2n-1)!}{2^{2n}(n!)^2} = \lg 2$$

$$(c) \sum_{n \geq 1} \left[\frac{(2n-1)!}{2^{2n}(n!)^2} \sum_{p=1}^{2n-1} \frac{1}{p} \right] = \frac{1}{2} \left(\frac{\pi^2}{3} + \lg^2 2 \right)$$

Proof. From the introduction, we know that :

$$\int_0^\pi \lg(1+r^2-2r \cos \theta) d\theta = 2\pi \lg|r|$$

that is $\forall r \in \mathbb{R}$; and $|r| > 1$. This implies :

$$\int_0^\pi \lg \left(1 - \frac{2r}{1+r^2} \cos \theta \right) d\theta = \pi \lg \frac{r^2}{1+r^2} ; |r| > 1.$$

The entire (or power) series of $\lg(1-x)$ is uniformly convergent within $|x| < 1$.

Therefore,

$$\forall r \in \mathbb{R}, \left| \frac{2r}{1+r^2} \right| < 1 \quad |r| \neq 1$$

implies through the uniform convergence theorem that :

$$-\sum_{n \geq 1} \frac{1}{n} \left(\frac{2r}{1+r^2} \right)^n \int_0^\pi \cos^n \theta d\theta = \pi \lg \frac{r^2}{1+r^2} ; |r| > 1.$$

Comparatively with steps developed in Proposition 2, using Wallis integrals, we deduce :

$$\begin{aligned} & -\sum_{n \geq 1} \frac{1}{2n} \left(\frac{2r}{1+r^2} \right)^{2n} 2 \frac{\pi}{2} \frac{(2n)!}{2^{2n}(n!)^2} = \pi \lg \frac{r^2}{1+r^2} \\ \Rightarrow & \sum_{n \geq 1} \frac{(2n)!}{2n(n!)^2} \left(\frac{r}{1+r^2} \right)^{2n} = \lg \left(1 + \frac{1}{r^2} \right) \quad \forall |r| > 1. \end{aligned}$$

Hence :

$$\sum_{n \geq 1} \frac{(2n)!}{2n(n!)^2} \left(\frac{r}{1+r^2} \right)^{2n} = \lg(1+r^2) \quad \forall |r| < 1.$$

This establishes (a)

In particular, as r tends to 1 we get the result **(b)** :

$$\sum_{n \geq 1} \frac{(2n-1)!}{2^{2n} (n!)^2} = \lg 2$$

From the same kernel of the logarithm analytic function, we have the Parseval identity (see the Introduction):

$$\int_0^\pi \frac{1}{4} \lg^2(1+r^2-2r \cos \theta) d\theta = \frac{\pi}{2} \sum_{n \geq 1} \frac{r^{2n}}{n^2} \quad ; |r| < 1.$$

Then, using the same preceding steps and the Taylor expansion of $\lg^2(1-x)$ in the disc $|x| < 1$, and making $r \rightarrow 1$; we obtain the series **(c)**:

$$\sum_{n \geq 1} \left[\frac{(2n-1)!}{2^{2n} (n!)^2} \cdot \sum_{p=1}^{2n-1} \frac{1}{p} \right] = \frac{1}{2} \left(\frac{\pi^2}{3} + \lg^2 2 \right)$$

In the same manner, we can compute for definite $n \in \mathbb{N}$, the coefficients:

$$|r| < 1, \int_0^\pi \lg(1+r^2-2r \cos \theta) \cos n\theta d\theta = -\pi \frac{r^n}{n}, \quad \forall n \in \mathbb{N}^*.$$

The series function $h(r)$ in (a) of Prop 3 is a generating function equivalent to the

Analytic $\lg(1+z^2)$, $|z| < 1$. More generating functions as $\lg^2(1-x)$, $Lg^3(1-x)$ are treated in details in Part II [2].

Proposition 4 : We have the following integrals:

$$(a) \int_0^\pi \lg^2(1+r^2-2r \cos \theta) d\theta = 2\pi \sum_{n \geq 1} \frac{r^{2n}}{n^2}, \quad \forall |r| < 1$$

$$(b) \int_{-\pi}^\pi \lg^2(1+r^2-2r \sin \theta) d\theta = 4\pi \sum_{n \geq 1} \frac{r^2}{n^2}, \quad \forall |r| < 1$$

$$(c) \int_{-\pi}^\pi \lg^2(1 \pm \cos \theta) d\theta = \int_{-\pi}^\pi \lg^2(1 \pm \sin \theta) d\theta = \frac{2}{3} \pi^3 + 2(\lg 2)^2$$

Proof.

(a) From the Fourier - Parseval kernel of the analytic function $\lg(1 \pm z)$ in the Disc $|z| < 1$ we deduce that (see Introduction) :

$$\int_0^\pi \lg^2(1+r^2-2r \cos \theta) d\theta = 2\pi \sum_{n \geq 1} \frac{1}{n^2} r^{2n}, \quad \forall |r| < 1$$

(b) To get the result as in (a), but depending upon sine function we use the analytic extension:

$$f(z) = \lg(1 \pm iz), \quad \forall |z| < 1$$

The Fourier - Parseval kernel in this case leads to:

$$\int_{-\pi}^\pi \lg^2(1+r^2-2r \sin \theta) d\theta = 4\pi \sum_{n \geq 1} \frac{1}{n^2} r^{2n}, \quad \forall |r| < 1.$$

(c) From the relation **(a)**, using Abel and Lebesgue's convergence theorems as $r \rightarrow 1$, we get :

$$\int_0^\pi \lg^2[2(1-\cos \theta)] d\theta = 2\pi \sum_{n \geq 1} \frac{1}{n^2} = \frac{1}{3} \pi^3$$

Developing the square and using the fact that:

$$\int_0^\pi \lg(1-\cos \theta) d\theta = -\pi \lg 2$$

obtained from the coefficient $a_0 = 0$ in the Fourier expansion of $\text{Re} [\lg(1-z)]$, we get the desired result. We remind (see Introduction) that:

$$\int_0^\pi \lg(1+r^2-2r \cos \theta) d\theta = 0, \quad \forall |r| < 1.$$

In the same way, the result depending upon sine function is proved, using the kernel relation in **(b)**.

Formulas such as **(c)** in Proposition 4 are well known under the compact form

$$\int_0^{\frac{\pi}{2}} \lg^2(\cos \theta) d\theta = \int_0^{\frac{\pi}{2}} \lg^2(\sin \theta) d\theta = \frac{1}{24} \pi^3 + \frac{\pi}{2} \lg^2 2$$

However, relations **(a)** and **(b)** in the same proposition show that general integrals do exist, depending upon parameter $r \in \mathbb{R}; |r| < 1$.

They are analytic and generating functions with respect to this variable and that they are of Fourier - Parseval kernel type.

Proposition 5 : We have the following integrals

$$(a) \int_0^\pi \lg(1+r^2-2r \cos \theta) \lg(1+r^2+2r \cos \theta) d\theta = \int_0^\pi \lg(1+r^2-2r \sin \theta) \lg(1+r^2+2r \sin \theta) d\theta = \frac{\pi}{2} \left[\sum_{n \geq 1} \frac{r^{4n}}{n^2} - 4 \sum_{n \geq 0} \frac{r^{2(2n+1)}}{(2n+1)^2} \right], \quad \forall |r| < 1$$

$$(b) \int_0^\pi \lg(1-\cos \theta) \lg(1+\cos \theta) d\theta = \int_0^\pi \lg(1-\sin \theta) \lg(1+\sin \theta) d\theta = -\frac{\pi^3}{6} + \pi(\lg 2)^2$$

Proof. (a) Through the determination of the Fourier - Parseval kernels of the analytic extension ;

$$\arctan z = \frac{1}{2i} \lg \left[\frac{1+iz}{1-iz} \right]; \quad \text{in the Disc } |z| < 1.$$

We deduce that:

$$\int_{-\pi}^\pi \lg^2 \left[\frac{1+r^2-2r \sin \theta}{1+r^2+2r \sin \theta} \right] d\theta = 16\pi \sum_{n \geq 0} \frac{r^{2(2n+1)}}{(2n+1)^2}; \quad |r| < 1.$$

$$\text{then: } \int_{-\pi}^\pi \lg^2 \left[\frac{1+r^2-2r \sin \theta}{1+r^2+2r \sin \theta} \right] d\theta$$

$$= \int_{-\pi}^{\pi} \lg^2(1+r^2-2r\sin\theta) d\theta + \int_{-\pi}^{\pi} \lg^2(1+r^2+2r\sin\theta) d\theta$$

$$+ \int_{-\pi}^{\pi} -2\lg(1+r^2-2r\sin\theta)\lg(1+r^2+2r\sin\theta) d\theta$$

Therefore, using **(b)** of Prop 4, we get :

$$\int_{-\pi}^{\pi} \lg(1+r^2-2r\sin\theta)\lg(1+r^2+2r\sin\theta) d\theta$$

$$= \frac{1}{2} \left[8\pi \sum_{n \geq 1} \frac{r^{2n}}{n^2} - 16\pi \sum_{n \geq 0} \frac{r^{2(2n+1)}}{(2n+1)^2} \right]$$

$$\Rightarrow \int_{-\pi}^{\pi} \lg(1+r^2-2r\sin\theta)\lg(1+r^2+2r\sin\theta) d\theta$$

$$= 4\pi \left[\sum_{n \geq 1} \frac{r^{2n}}{n^2} - 2 \sum_{n \geq 0} \frac{r^{2(2n+1)}}{(2n+1)^2} \right] = \pi \left[\sum_{n \geq 1} \frac{r^{4n}}{n^2} - 4 \sum_{n \geq 0} \frac{r^{2(2n+1)}}{(2n+1)^2} \right]$$

Hence:

$$\int_0^{\pi} \lg(1+r^2-2r\sin\theta)\lg(1+r^2+2r\sin\theta) d\theta$$

$$= \frac{\pi}{2} \left[\sum_{n \geq 1} \frac{r^{4n}}{n^2} - 4 \sum_{n \geq 0} \frac{r^{2(2n+1)}}{(2n+1)^2} \right]$$

for all $r \in \mathbb{R}$ and $|r| < 1$

In a similar manner, using the analytic function

$$\arctan(iz) = \frac{1}{2i} \lg \left[\frac{1-z}{1+z} \right] \text{ for } |z| < 1.$$

and the property **(a)** of Proposition 4, we get the same result depending up on cosine function.

Finally, a passage to the limit when r tends to 1 gives the property **(b)** and the proof is complete.

Throughout the different steps of our study, we did not consider the kernels $\text{Im}f(re^{i\theta})$, which are of the type $\arctan(r\cos\theta, r\sin\theta)$ and the Fourier - Parseval properties related to them. More details may be found in [1] and [2].

As an example, we have

$$\int_{-\pi}^{\pi} \left[\arctan \frac{r \cos \theta}{1+r \sin \theta} \right]^2 d\theta = \pi \sum_{n \geq 1} \frac{1}{n^2} r^{2n} \quad ; \quad |r| < 1$$

and

$$\int_{-\pi}^{\pi} \left[\arctan \frac{\cos \theta}{1+\sin \theta} \right]^2 d\theta = \frac{\pi^3}{6}$$

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