Solving Non Separable Convex Quadratic Programming Problems

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Abstract: *The aim of this paper is to present a new method for solving non-separable quadratic problems. In a first step we transform the non-separable quadratic problem in a separable quadratic problem equivalent. In a second step we solve the quadratic problem separable by the method of projection. The principle of this method is to calculate the critical point, if it is a feasible solution then this is the optimal solution. Otherwise, we construct a new feasible set by a homographic transformation on which we project the transformed critical point and we give the optimal solution belonging to the feasible set of the original problem. Note that the resolution is done directly on the primal separable quadratic problem and not on the linear problem as do several methods.*
The method is purely analytical and avoids the thorny problem of the choice of the initial solution.

Keyword: *Non Separable Quadratic Programming, Concave maximizing, Eigen values, Projection Method, Homographic Transformation.*

I. Introduction

The separable quadratic programming (Stephen B. and V. Lieven , 2004.) is very important in industry and finance. Non separable problems are traditionally solved by linear programming techniques (Hillier and Lieberman , 2001). In some others cases an allocation problem is formulated as a nonlinear constrained optimization problem and solved by a quadratic programming method (Gill et al. , 2002). Other approaches (Friedlander & all. 2012), prefer the method of semi- interior, and the method can be interpreted as an adjustment to the proximal point of primal-dual problems. The convergence problem has been studied in several articles including (Delbos F. Gilbert and J. Ch , 2003). On the other uses the modified Lagrange method (S. Ketabchi all . & 2009).

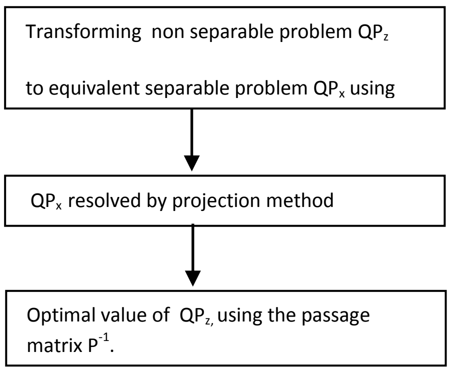
This article describes a new method based on the transformation of a non-separable quadratic programming problem in a separable equivalent problem. This coordinate transformation uses the Gauss pivot method to make the diagonal matrix representing the quadratic term of the objective function. Once we got equivalent problem, we apply it our quadratic programming algorithm based on the projection method.

Problem Formulation

The Matrix form of our non-separable problem is:

$$(\mathrm{QP}_z)\begin{cases}\max \;\; f(z) = c'z - z'Hz \\ Az \le b \\ 0 \le z \le u\end{cases} .$$

The following schema shows the steps of resolution:



Before making a change of variables based on the diagonaliza-tion of the matrix H, we recall the basics of the diagonalization.

II. TRANSFORMING NON SEPARABLE PROGRAMMING

Definition (Eigenvalues , eigenvectors)
A scalar λ is an eigenvalue of the matrix H if and only if there exists a vector $v \neq 0_{R^n}$, $Hv = \lambda v$; v is called an eigenvector of the matrix associated with the eigenvalue λ .

Theorem 1:

1. *Let H a matrix $n \times n$. λ is an eigenvalue of H if and only if* $\det\left(H - Id_{(n,n)}\right) = 0_{R^n}$.

2. *If λ is an eigenvalue of H then solution $v \neq 0_{R^n}$ of $\det\left(H - Id_{(n,n)}\right) = 0_{R^n}$ is an eigenvector associated with λ.*

Theorem 2:

Let H a symmetric matrix $n \times n$. Then there exists an orthogonal matrix P which diagonalizes H

We give the theorem which transforms a quadratic form with cross terms (in a quadratic form with only squared terms $P'HP = D$).

Theorem 3:

Let $H = \left(q_{ik}\right)_{n\times n}$ a symmetric matrix of eigenvalues $\lambda_1,....\lambda_n$ and P an orthogonal matrix that diagonalizes H. Then the change in the coordinate $z = Px$ transforms $\sum_{i,k} q_{ik} Z_i Z_k$ to $\sum_i \lambda_i x_i^2$.

The matrix P is the matrix of passage is the matrix consisting of the orthonormal eigenvectors associated with eigenvalues of H .

$$\left(\mathrm{QP_x}\right) \begin{cases} \max\ g(x) = cPx - x'Dx \\ APx \leq b \\ 0 \leq Px \leq u \qquad x \in X \end{cases}$$

where,

- $D = P'HP$ is a diagonal matrix consisting of positive real eigenvalues of the matrix H.
- X is a finite set of values resulting from the product of the matrix P and the vector u .

We transformed quadratic function f to canonical quadratic function g and then non-separable quadratic problem $\left(\mathrm{QP_Z}\right)$ is transformed to separable quadratic problem $\left(\mathrm{QP_x}\right)$.

The next step solves separable quadratic problem $\left(\mathrm{QP_x}\right)$.

III. SOLVING SEPARABLE QUADRATIC PROGRAMMING

A. *Projection Method*

For simplicity, we write the canonical separable quadratic problem

$$\left(\mathrm{QP_x}\right) = \begin{cases} \max_{x\in\Omega}\ g(x) = \sum_{i=1}^{n} \alpha_i x_i + \beta_i x_i^2 \\ \alpha_i \in R, \quad \beta_i < 0 \\ \Omega = \left\{ \ x \in R^n : 0 \leq x \leq u \ and \ Ax \leq b \ \right\} \end{cases}$$

Let $x^* = (x_i^*)_i = \left(\dfrac{-\alpha_i}{2\beta_i}\right)_i$;

$$y^* = (y_i^*)_i = \left(\frac{\alpha_i}{2\sqrt{-\beta_i}}\right)_i$$

$$y = (y_i)_i = \left(\sqrt{-\beta_i}\, x_i\right)_i;$$

and

$$\underset{x\in\Omega}{Max}\, g(x) = g(\bar{x}); \ \ \bar{x} = (\bar{x}_i)_i \in \Omega.$$

The following theorem proves the algorithm of projection method described in this paper.

Theorem 4 : There exists a closed bounded convex set Ω' of R^n, and a vector $y_0 = (y_{0i})_i \in \Omega'$, such that the following statements are satisfied:

1. $\underset{x\in\Omega}{Max}\, g(x) = g\left(x^*\right) - \left\|y^* - y_0\right\|^2;$
2. $\left\|y^* - y_0\right\| = \inf_{y\in\Omega'} \left\|y^* - y\right\|$;
3. $\bar{x}_i = \dfrac{y_{0i}}{\sqrt{-\beta_i}} \quad \forall i$.

Proof:

For every $x \in \Omega,$ let $\Delta g_i = \left(\alpha_i x_i^* + \beta_i x_i^{*2}\right) - \left(\alpha_i x_i + \beta_i x_i^2\right).$ Then

$$\Delta g_i = \alpha_i \left(\frac{-\alpha_i}{2\beta_i}\right) + \beta_i \left(\frac{-\alpha_i}{2\beta_i}\right)^2 - \alpha_i x_i - \beta_i x_i^2$$

$$= \beta_i \left(x_i + \frac{\alpha_i}{2\beta_i}\right)^2 = \beta_i \left(x_i - x_i^*\right)^2.$$

But $\quad g\left(x^*\right) - g(x) = \sum_{i=1}^{n} \Delta g_i$

$$g\left(x^*\right) - g(x) = \sum_{i=1}^{n} -\beta_i \left(x_i - x_i^{\bullet}\right)^2$$ for all $x \in \Omega$,

Therefore,

$$\inf_{x\in\Omega}\left(g(x^*)-g(x)\right)=\inf_{x\in\Omega}\left(\sum_{i=1}^{n}-\beta_i\left(x_i-x_i^{\bullet}\right)^2\right)$$

can be written in the following form:

$$g(x^*)-\underset{x\in\Omega}{Max}\,g(x)=\inf_{x\in\Omega}\sum_{i=1}^{n}\left(\sqrt{-\beta_i}\left(x_i^*-x_i\right)\right)^2$$

$$=\inf_{y\in\Omega'}\sum_{i=1}^{n}\left(y_i^*-y_i\right)^2.$$

Let $\Omega' = \{\ y=(y_i)_i \in R^n : y_i=\sqrt{-\beta_i}\,x_i, x=(x_i)_i$, because

$$\inf_{y\in\Omega'}\sum_{i=1}^{n}\left(y_i^*-y_i\right)^2=\left\|y^*-y_0\right\|^2, y_0\in\Omega'.$$

Thus $\underset{x\in\Omega}{Max}\,g(x)=g(x^*)-\left\|y^*-y_0\right\|^2$, hence property (1).

Because $\inf_{y\in\Omega'}\left\|y^*-y\right\|^2=\left\|y^*-y_0\right\|^2$, then $\left\|y^*-y_0\right\|^2\le\left\|y^*-y\right\|^2$, for every $y\in\Omega'$.

This implies that $\left\|y^*-y_0\right\|\le\left\|y^*-y\right\|$, for every $y\in\Omega'$. We have therefore $\left\|y^*-y_0\right\|\le\inf_{y\in\Omega'}\left\|y^*-y\right\|$.

Because $y\in\Omega'$ then $\left\|y^*-y_0\right\|\ge\inf_{y\in\Omega'}\left\|y^*-y\right\|$; then $\left\|y^*-y_0\right\|=\inf_{y\in\Omega'}\left\|y^*-y\right\|$. Hence property (2).

The vector y_0 is the projection of the vector y^* onto the new convex Ω'. We have

$$\underset{x\in\Omega}{Max}\,\varphi(x)=g(\bar{x})=g(x^*)-\left\|y^*-y_0\right\|^2$$

$$=g(x^*)-\inf_{x\in\Omega}\sum_{i=1}^{n}\left(y_i^*-\sqrt{-\beta_i}\,x_i\right)^2$$

$$=g(x^*)-\sum_{i=1}^{n}\left(y_i^*-\sqrt{-\beta_i}\,\bar{x}_i\right)^2.$$

Hence property (3).

The transformation $T:\Omega\subset R^n\to\Omega'\subset R^n$, for each $x\in\Omega$, associating $T(x)=\Lambda x,\ \Lambda=\left(\sqrt{-\beta_1},......,\sqrt{-\beta_n}\right)$ has as Jacobean matrix

$$\begin{bmatrix}\sqrt{-\beta_1} & 0 & \dots & 0\\ 0 & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots\\ 0 & \dots & 0 & \sqrt{-\beta_n}\end{bmatrix}$$

Its determinant is $\prod_{i=1}^{n}\sqrt{-\beta_i}\neq 0$. Then it is conform.

Algorithm of computing the optimal solution of $(\mathrm{QP_x})$.

Algorithm

Initialization: matrix A, vectors b, c, alpha and beta.

__If__ all β_i = -1 **then** Ω'= Ω __**else**__ build Ω' and compute the critical point

__**If**__ $x^*\in$ Ω **then** x^* is the Optimal solution. **STOP**.

__**else**__

begin

for i= 1 to n

__**begin**__

$$y^*=(y_i^*)_i=\left(\frac{\alpha_i}{2\sqrt{-\beta_i}}\right)_i$$

$$y_0=P_\Omega\left(y^*\right)=y^*-\frac{<y^*,a>-b}{\|a\|^2}a$$

$$\bar{x}_i=\frac{y_{0i}}{\sqrt{-\beta_i}}$$

end

__**If**__ $\bar{x}\in\Omega$ then $\bar{x}$ is the Optimal solution ; compute $g(\bar{x})$ __**STOP**__.

__**else**__

change the supporting hyper plane separator

end.

Example

$$(\mathrm{Q}P_z)\begin{cases}\max\ f(z)=69z_1+71z_2-15z_1^2-17z_2^2-2z_1z_2\\ 81z_1+50z_2\le 61\\ 17z_1+2z_2\le 105\\ 0\le z_1\le 3\\ 0\le z_2\le 2\end{cases}$$

It can be written as

$$H = \begin{bmatrix} -15 & -1 \\ -1 & -17 \end{bmatrix} \qquad c = \begin{bmatrix} 69 & 71 \end{bmatrix}$$

$$A = \begin{bmatrix} 81 & 50 \\ 17 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 61 \\ 105 \\ 3 \\ 2 \end{bmatrix}.$$

Begin by transforming the non-separable problem QP_z in separable problem QP_x.

The diagonal matrix is

$$D = \begin{bmatrix} -17.4142 & 0 \\ 0 & -14.5858 \end{bmatrix}$$

and the transition matrix is

$$P = \begin{bmatrix} 0.3827 & -0.9239 \\ 0.9239 & 0.3827 \end{bmatrix}$$

the constraint matrix A is transformed into

$$A' = A * P = \begin{bmatrix} 77.1913 & -55.7001 \\ 8.3534 & -14.9406 \\ 0.3827 & -0.9239 \\ 0.9239 & 0.3827 \end{bmatrix}.$$

The vector c is transformed into $c'P = (92.0006 \quad -36.5772)$

$$(QP_x)\begin{cases} \max\ g(x) = c'Px - x'Dx \\ APx \le b \\ 0 \le Px \le u \qquad (x \in X) \end{cases}$$

And so we have to solve a separable quadratic programming

$$(QP_x)\begin{cases} \max g(z) = 92.0006x_1 - 36.5772x_2 - 17.4142x_1^2 \\ 77.1913x_1 \quad -55.7001x_2 \le 61 \\ 8.3534x_1 \quad -14.9406x_2 \le 105 \\ 0.3827x_1 \quad -0.9239x_2 \le 3 \\ 0.9239x_1 \quad 0.3827x_2 \le 2 \end{cases}$$

The critical point is

$$x^* = \begin{cases} \dfrac{\partial f}{\partial x_1}(x_1, x_2) = -1.4905 \\ \dfrac{\partial f}{\partial x_2}(x_1, x_2) = 2.7617 \end{cases}, \qquad x^* \notin \Omega,$$

because $x^*(1) < 0$

It is therefore necessary to construct Ω' transformed Ω by the transformation T.

$$\Omega' = \{(x_1, x_2) \in R^2 / A2x \le b \qquad avec \qquad A2 = A/sqrt(-D)\}$$

$$A2 = A/sqrt(-D) = \begin{bmatrix} 18.4977 & -14.5845 \\ 2.0018 & -3.9120 \\ 0.0917 & -0.2419 \\ 0.2214 & 0.1002 \end{bmatrix}$$

$$Y^* = (11.0232\ ,\ -4.78867)$$

is the transformed of the critical point x^*.

We recall the formula projection of a point y^* on a hyperplane $ay + b = 0$:

$$y_0 = P_{\Omega'}(y^*) = y^* - \frac{< y^*, a > -b}{\|a\|^2} a$$

Hyperplanes are here constraints of our problem,

therefore $y_0 = P_{\Omega'}(y^*) = (3.9310 \quad 0.8032)$.

The transition to $\bar{x}$ the optimal solution of the initial feasible set Ω is:

$$(\bar{x}_i)_i = \left(\frac{y_{0i}}{\sqrt{-\beta_i}}\right)_i = (0.9420 \quad 0.2103)$$

The value of the optimal solution to our original problem (QP_z) is:

$$x_{opt} = \bar{x} \times P' = (0.9420 \quad 0.2103) \times \begin{pmatrix} 0.3827 & 0.9239 \\ -0.9239 & 0.3827 \end{pmatrix}$$

$$= (0.1662 \quad 0.9508)$$

$$\boxed{f(x_{opt}) = 62.8742}.$$

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