SOME GENERALIZATIONS OF INTEGRAL INEQUALITIES FOR WEIGHTED HARDY OPERATORS WITH $0 < p < 1$

Senouci Abdelkader[1], Zanou Abdelkader[2],
[1]Departement of Mathematics, Laboratory of Informatics and Mathematics, University of Tiaret, Algeria
[2]Faculty of Applied Sciences, Laboratory of Informatics and Mathematics, University of Tiaret, Algeria
[1]kamer295@yahoo.fr
[2]zanou1985@gmail.com

Abstract- In this work we give some generalization of the results established and proved by A.Senouci and al (See [1] and [2])..

***Index Terms*—Inequalities, Hardy's operators.**

I. INTRODUCTION

LET f be a Lebesgue measurable function on $(0,+\infty)$, and let w denote a weight function on $(0,\infty)$ (that is a non-negative Lebesgue measurable function). For $0<p<1$, the weighted space function $L^p_w(0,\infty)$ is the space of all real-valued Lebesgue measurable functions with finite quasi-norm.

$$\|f\|_{L^p_w(0,\infty)} = \left(\int_0^\infty |f(x)|^p w(x)dx\right)^{\frac{1}{p}}.$$

The weighted Hardy operator is defined by

$$(H_w f)(x) = \frac{1}{W(x)}\int_0^x f(t)w(t)dt,\ x>0,$$

where $0<W(x)=\int_0^x w(t)dt<\infty$ for all $x>0$. Note that for $w(t)=1$, the operator H_w is the usual Hardy operator

$$(Hf)(x) = \frac{1}{x}\int_0^x f(t)dt.$$

. In [1] the following lemma was proved .

Lemma 1 *Let $0<p<1$, $c_1>0$, $A>0$, w be a weight function on $(0,\infty)$ such that $w(x)\le cw(y)$ for $0<y<x<\infty$. If f is a non-negative Lebesgue measurable function on $(0,\infty)$ such that for almost all $0<t<\infty$,*

$$f(t) \le A\left(\int_0^t w(y)y^{p-1}dy\right)^{\frac{-1}{p}}\left(\int_0^t f^p(y)w(y)y^{p-1}dy\right)^{\frac{1}{p}}, \tag{1}$$

then for all $x>0$

$$(H_w f)(x) \le \frac{c_2}{x.w^{\frac{1}{p}}(x)}\left(\int_0^x f^p(y)w(y)y^{p-1}dy\right)^{\frac{1}{p}}, \tag{2}$$

where $c_2 = p^{\frac{1}{p}}A^{1-P}c_1^{\frac{2}{p}-1}$.

Lemma 2 *Let $0<p<1$, $B>0$, w be a weight function on $(0,\infty)$ such that for all $x>0$, $\int_0^x w(t)dt<\infty$. If f is a non-negative Lebesgue measurable function on $(0,\infty)$ such that for almost all $0<x<\infty$*

$$\int_x^\infty f^p(y)w(y)y^{p-1}dy<\infty,$$

and

$$f(x) \le \frac{B}{x}\left(\int_x^\infty f^p(y)w(y)y^{p-1}dy\right)^{\frac{1}{p}}\left(\int_0^x w(y)dy\right)^{\frac{1}{1-p}} w^{\frac{1}{1-p}}(x), \tag{3}$$

Then for $r>0$

$$(H_w^* f)(r) \le pB^{1-P}w(r)\left(\int_r^\infty f^p(y)w(y)y^{p-1}dy\right)^{\frac{1}{p}}. \tag{4}$$

In [2] the following theorem was proved.

Theorem 1 *Let $0<p<1$, $B>0$, $x>0$ and $-\frac{1}{p}<\alpha<1-\frac{1}{p}$. If f is a non-negative Lebesgue measurable function on $(0,\infty)$ and satisfying condition (4), then*

$$\|\tau^\alpha(H^* f)(\tau)\|_{L^p(0,\infty)} \le pB^{1-p}(\alpha p+1)^{-\frac{1}{p}}\|y^{\alpha+1}f(y)\|_{L^p(0,\infty)}. \tag{5}$$

the aim of this work is to generalize some results obtained in [2].

II. MAIN RESULTS

Let f be a Lebesgue measurable function on $(0,\infty)$. The following theorem is the generalization of theorem 1 for weighted Hardy operator.

Theorem 2 *Let $0<p<1$, $x>0$, $B>0$, w be a weight function such that $w(r)<w(y)<\infty$ for $r<y<\infty$, and $-\frac{1}{p}<\alpha<1-\frac{1}{p}$. If f is non-negative Lebesque measurable function on $(0,\infty)$ satisfying condition (4), then*

$$\left\|r^\alpha(H_w^* f)(r)\right\|_{L^p(0,\infty)} \le pB^{1-p} \tag{6}$$

$$(p\alpha+1)^{-\frac{1}{p}}\left\|y^{\alpha+1}f(y)w^{\frac{p+1}{p}}(y)\right\|_{L^p(0,\infty)}. \tag{7}$$

Proof-
By (4) it follows that

$$\left\|r^\alpha(H_w^* f)(r)\right\|_{L^p(0,\infty)} \le pB^{1-P}\left\|r^\alpha w(r)\right.$$

$$\left.\left(\int_r^\infty f^p(y)w(y)y^{p-1}dy\right)^{\frac{1}{p}}\right\|_{L^p(0,\infty)}.$$

Let

$$I = pB^{1-P}\left\| r^{\alpha p}w^p(r)\left(\int_r^\infty f^p(y)w(y)y^{p-1}dy\right)\right\|_{L^1(0,\infty)}^{\frac{1}{p}}.$$

By fubuni theorem we have

$$I = pB^{1-P}\left\| f^p(y)w(y)y^{p-1}\left(\int_0^y r^{\alpha p}w^p(r)dr\right)\right\|_{L^1(0,\infty)}^{\frac{1}{p}}.$$

Hence

$$\begin{aligned} I &\le pB^{1-P}\left\| f^p(y)w(y)w^p(y)y^{p-1}\left(\int_0^y r^{\alpha p}dr\right)\right\|_{L^1(0,\infty)}^{\frac{1}{p}} \\ &= pB^{1-P}\left\| f^p(y)w(y)w^p(y)y^{p-1}\left(\int_y^y r^{\alpha p}dr\right)\right\|_{L^1(0,\infty)}^{\frac{1}{p}} \\ &= pB^{1-P}\left\| f^p(y)w(y)w^p(y)y^{p-1}\left[\frac{r^{\alpha p+1}}{\alpha p+1}\right]_0^y\right\|_{L^1(0,\infty)}^{\frac{1}{p}} \\ &= pB^{1-P}\left\| f^p(y)w(y)w^p(y)y^{p-1}\frac{y^{\alpha p+1}}{\alpha p+1}\right\|_{L^1(0,\infty)}^{\frac{1}{p}} \\ &\le pB^{1-P}\left\| f^p(y)y^{\alpha p+1}y^{p-1}\frac{w^{p+1}(y)}{\alpha p+1}\right\|_{L^1(0,\infty)}^{\frac{1}{p}} \\ &= p^{1-\frac{1}{p}}B^{1-p}(\alpha+\frac{1}{p})^{-\frac{1}{p}}\left\| w^{\frac{p+1}{p}}(y)y^{\alpha+1}f(y)\right\|_{L^p(0,\infty)}. \end{aligned}$$

Remark 1 *If we put $w(x)=1$ in (**??**) we get Theorem* 1.

Now we lead with the operator $\frac{1}{x}\int_x^\infty f(t)dt$ where $f(x)\le \frac{M}{x}\left(\int_x^\infty f^p(t)t^{p-1}dt\right)^{\frac{1}{p}}$.

Lemma 3 *Let $0<p<1$, $M>0$ and $x>0$. If f is a non-negative Lebesgue measurable function on $(0,\infty)$ such that for almost all $0<x<\infty$,*

$$\int_x^\infty f^p(t)t^{p-1}dt<\infty$$

and

$$f(x)\le \frac{M}{x}\left(\int_x^\infty f^p(t)t^{p-1}dt\right)^{\frac{1}{p}}. \qquad (8)$$

Then

$$\left(\int_x^\infty f(t)dt\right)^p \le K\int_x^\infty f^p(t)t^{p-1}dt, \qquad (9)$$

where $K=p^pM^{P(1-p)}$.

Proof-
By (8) for $t>0$ it follows that

$$f^{1-p}(t)\le M^{1-p}t^{p-1}\left(\int_t^\infty f^p(y)y^{p-1}dy\right)^{\frac{1-p}{p}}.$$

Hence

$$f(t)\le M^{1-p}f^p(t)t^{p-1}\left(\int_t^\infty f^p(y)y^{p-1}dy\right)^{\frac{1-p}{p}},$$

$$= pM^{1-p}(-1)\left[\left(\int_t^\infty f^p(y)y^{p-1}dy\right)^{\frac{1}{p}}\right]'.$$

Integrating over (x,∞) we obtain

$$\int_x^\infty f(t)dt$$

$$\le pM^{1-p}\lim_{c\to\infty}\left(\left(\int_x^\infty f^p(y)y^{p-1}dy\right)^{\frac{1}{p}} - \left(\int_c^\infty f^p(y)y^{p-1}dy\right)^{\frac{1}{p}}\right),$$

$$\le pM^{1-p}\left(\int_x^\infty f^p(y)y^{p-1}dy\right)^{\frac{1}{p}},$$

hence

$$\left(\int_x^\infty f(t)dt\right)^p \le p^pM^{p(1-p)}\int_x^\infty f^p(y)y^{p-1}dy. \qquad (10)$$

Theorem 3 *Let $0<p<1$, $x>0$ and $\alpha>1-\frac{1}{p}$. If f is non-negative Lebesgue measurable function on $(0,\infty)$ and satisfies (8) for all $x>0$, then*

$$\|x^\alpha(\widetilde{H}f)(x)\|_{L^p(0,\infty)}\le K_1\|t^\alpha f(t)\|_{L^p(0,\infty)}, \qquad (11)$$

where $(\widetilde{H}f)(x)=\frac{1}{x}\int_x^\infty f(t)dt$ and $K_1=p^{1-\frac{1}{p}}(\alpha-1+\frac{1}{p})^{-\frac{1}{p}}M^{1-p}$.

Proof-
Let $(\widetilde{H}f)(x)=\frac{1}{x}\int_x^\infty f(t)dt$, $0<x<t<\infty$ and

$$J=\|x^\alpha(\widetilde{H}f)(x)\|_{L^p(0,\infty)}.$$

By definition of J we have

$$J=\left[\int_0^\infty x^{p\alpha}(\widetilde{H}f)^p(x)dx\right]^{\frac{1}{p}},$$

$$=\left[\int_0^\infty x^{p(\alpha-1)}\left(\int_x^\infty f(t)dt\right)^p dx\right]^{\frac{1}{p}}.$$

By (10) is follows that:
$\left[\int_0^\infty x^{p(\alpha-1)}\left(\int_x^\infty f(t)dt\right)^p dx\times\right]^{\frac{1}{p}}$
$\le pM^{1-p}\left[\int_0^\infty x^{p(\alpha-1)}\times\left(\int_x^\infty f^p(t)t^{p-1}dt\right)dx\right]^{\frac{1}{p}}$,
by Fubuni theorem and $\alpha>1-\frac{1}{p}$, we obtain

$$\left[\int_0^\infty x^{p(\alpha-1)}\left(\int_x^\infty f^p(t)t^{p-1}dt\right)dx\right]^{\frac{1}{p}} =$$

$$\left[\int_0^\infty f^p(t)t^{p-1}\left(\int_0^t x^{p(\alpha-1)}dx\right)dt\right]^{\frac{1}{p}},$$

$$=p^{-\frac{1}{p}}\left(\alpha-1+\frac{1}{p}\right)^{-\frac{1}{p}}\left[\int_0^\infty f^p(t)t^{p-1}t^{p(\alpha-1)+1}dt\right]^{\frac{1}{p}}.$$

Hence

$$\left[\int_0^\infty x^{p(\alpha-1)}\left(\int_x^\infty f(t)dt\right)^p dx\right]^{\frac{1}{p}}$$

$$\le M^{1-p}p^{1-\frac{1}{p}}\left(\alpha-1+\frac{1}{p}\right)^{-\frac{1}{p}}\left[\int_0^\infty f^p(t)t^{p\alpha}dt\right]^{\frac{1}{p}}.$$

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