Afak For Science Journal
Volume: 05 / N°: 04(2020), P .347-361

Received 19/07/2020 Accepted 15/08/2020

On the Geometric Structures of Compact Manifold (Distance Function, Critical Points On the Homology Representation)

Khalid Abd Assalam Ateia Ismaeel *

khalidateia@yahoo.com

Red Sea Uniersity,

(Sudan)

Abstract: In the field of algebraic topology, the homology of an $n-dimensional$ space reflects the geometric properties in the interior of such space. We found that and with appropriate parameter ε_n, we can built a union of optimizes neighborhoods to represents geometric structure in an $n-dimensional$ manifolds. Upon the notion of the critical points of the distance function, we can generate an abelian group, which represents a basis for such homology.

Key words: Simplicial complex, excursion sets, distance function, critical points

* Corresponding author

1- Introduction:

Topology is a mathematical field which studies the qualitative geometric information. Such branch of mathematics had interested in the behavior of functional, than its qualitative values and diacritics on intrinsic geometric properties of objects. We had many reasons to deal with topology:

- The topology deals with accuracy geometric information such as connected component of spaces (Homology).
- The topology studies the closeness of point each to other under the notion of subsets (clustering)
- Topology studies the intrinsic geometric properties of such sets with independence of chosen coordinates

In the field of algebraic topology, we look at the relation between topological spaces and its algebraic structure upon classifying these spaces into similar spaces and study their geometrical properties and the behavior of mappings between them. In such field of studies we care about assigning the algebraic group, their appropriate geometric structure. The homology, co – homology, and homotopy groups can be used in such classifying [10, 13].

By the other hand, in the area of manifold learning we interested on the geometric concept of the cloud of data points distributed randomly on $\mathbb{R}^N$. In such topic, we can follow the technique of concentration of measure phenomenon. In such technique we had to embed this data with an isometric map, to a legally Euclidean space to get the geometric information. The other effective technique, works on deriving the homology, co – homology of such data. And generate a homotopy group to classifying the collection of observations (data points) into complexes of similar shapes [3, 6, 10, 11, 13]. Under the persistent homology we can track how to reshape the collection of data points and represent its geometric properties upon topological invariants [9, 13, 14]. A qualitative property (invariants) of a topological space X is to study their decomposition into path – connected components.

1.1 Definition [3]: For any topological space$X$, abelian group A, and integer $k \geq 0$, there is assign group $H_k(X, A)$.
1.2 Definition [3]: An abstract simplicial complex is a pair $(X, \sum\ )$, where X is a finite set, and $\sum$ is a family of nonempty subset of X such that if $\sigma \in \sum$, $\tau \subseteq \sigma$ implies that $\tau \in \sum$. Associated to a simplicial is a topological space $|(X, \sum)|$, which may be defined using a bijection $\varphi: X \to \{1, 2, \dots, N\}$ as a subspace of $\mathbb{R}^N$ given by the union $\cup_{\sigma \in \sum} c(\sigma)$, where $c(\sigma)$is the convex hull of the set $\{e_{\varphi(s)}\}_{s \in \sigma}$, where e_i denotes the i - th standard basis vector.

Let $M \subseteq \mathbb{R}^N$be a compact manifold, and let $f: M \to \mathbb{R}^N$ be smooth function (bijective) and let $f(M) \longmapsto \{x_1, \dots, x_n\} \in \mathbb{R}^N$. The notion of excursion sets $f^{-1}([\varepsilon, +\infty))$, which is called level set for the function f, connects the global properties of sub sets of M in the way that $\left(M, f^{-1}([\varepsilon, +\infty))\right)$ is a simplicial complex. So we can deal with $\left|\left(M, f^{-1}([\varepsilon, +\infty))\right)\right|$ as a topological space. the $\bigcup_{x_i \in \{x_1,\ldots,x_n\}} f_{x_i}^{-1}([\varepsilon, +\infty))$ is homotopy equivalent to M [4, 15, 16]. The most advantage of the excursion sets:

$$f^{-1}([\varepsilon, +\infty)) = \{x_i \in M: f(x_i) \geq \varepsilon\}, x_i \in \{x_1, \ldots, x_n\} \ldots (1)$$

is that, there is no change on the internal geometry of the data shape happens with in the level of the set (ε). So, it preserves the geometry of the class of data point with in $\varepsilon - range$ [9, 10, 12]

Backing to the field of points $\{x_1, \ldots, x_n\} \in \mathbb{R}^N$, Adler [15] take such field as a critical points of the embedding function $f: M \to \mathbb{R}^N$. Morse's Theorem [16] had discussed this situation with Euler Characteristic functional (EC). The critical points of the distance function formulate an appropriate homology which is equivalent to M. From statistically point of view the critical points of the distance function form an $i.i.d$ points which is deformation track the homology of M [10, 11, 12]. It behaves as random variables of Gaussian kind. With such points we can built an excursion set which is homotopy equivalent to M.

Our 2nd section takes the notion of neighborhood as criteria to represent the Geometric information of collection of points using the concept of simplicial complexes. The 3rd section consider the homology of a collection of data points to describe the geometry of an $n - dimensional\, manifold$. The fourth section treat the notion of distance function and its sublevel set which is called excursion set as a homology to describe the algebraic geometry on high – dimensional spaces. The fifth section deals with the critical points of the distance function as النواة to build the excursion set. Lastly we had a brief discussion.

2- Neighborhood and Geometry of Manifolds:

The importance of analyzing and studying the techniques of estimating geometry on high – dimensional spaces takes a big realm in many areas of mathematics. To get an inner – structure of an $n - dimensional\, manifolds$ we need to study it's locally topology and fetch its neighborhood and it's legally homeomorphic space.

2.1 Definition [7]: Let X be a pre – compact path metric space. For $\varepsilon > 0$ the $\varepsilon - capacity$ of X, denoted $Cap_\varepsilon(x)$, is the minimum number of $radius - \varepsilon$ balls required to cover X. In other words, $Cap_\varepsilon(x)$ is the minimum number of points in an $\varepsilon - net$ in X (An $\varepsilon - net$ is a subset N of X such that for each $x \in X, d(x, N) < \varepsilon$).

The $\varepsilon - net$ in X will represent the usual neighborhood under the notion "nearest", which cover an $n - dimensional\, manifold$.

2.2 Theorem [2]: Let C be a nonempty closed convex subset of a Hilbert space H (particularly, of the Euclidean space $\mathbb{R}^N$). Then, for each $x \in H$ there is, and is unique, a point $p_c(x)$ of C such that

$$\|x - p_c(x)\| = d(x,c) \dots (2)$$

That is, there exists a unique nearest point of C to x.

So, we need to estimate a legally system to create appropriate covering on an $n-manifold$.

2.3 Definition [17]: An "approach space" is a pair $(X, \mathcal{G})$ where X is an arbitrary set and $\mathcal{G}$ is an approach distance on an approach system on X.

Such approach system in Definition (2.3) above generates a topology which characterize the interior geometry of an $n - dimensional\ manifold$.

2.4 Theorem [17]: (approach distance & approach system)

1- If δ is a distance on X, then

$$A_\delta(x) = \{\varphi \in [0,\infty)^X, \forall A \in 2^X: inf_{a\in A}\varphi(a) \leq \delta(x,A), x \in X\} \dots (3)$$

Defines an approach system on X.

2- If $A = (A(x))_{x\in X}$ is an approach system on X, then:

$$\delta_A(x,A) = sup_{\varphi\in A(x)} inf_{a\in A}\varphi(a), x \in X, A \in 2^X \dots (4)$$

Where 2^X stands for the power set of X.

Then, we can conclude that

$$d_A := inf_{\substack{a\in A\\ x\in X}}\|x - a\| \dots (5)$$

Will defines an approach distance function to track the geometry on an $n - manifold\ M$.

2.5 Definition [17]: Let $(X, \|.\|)$ be a normed space. Then $D_{(X,X')}$ is a collection of $\infty_p - metrics$ on X which is saturated for the formation of finite suprema. It therefore generates an "approach distance" on X, which will denoted by $\delta_{(X,X')}$ and we will call the "weak approach distance", given by the following formula:

$$\delta_{(X,X')}: X \times 2^X \to [0,\infty) \dots (6)$$

$$(x,A) \to sup_{\varphi\in 2_0^{(B^*_{X'})}} inf_{a\in A} p_\varphi(x-a) \dots (7)$$

2.6 Proposition: Let $(X, \|.\|)$ be a normed space. Then the $\infty_p - metric$ core flection of $(X, \delta_{(X,X')})$ is $(X, d_{\|.\|})$.

The supporting domain of such distance function will be a guarantee to build local topology.

2.7 Proposition: (Convexity of level sets)

Let f be a convex function with the domain M. Then, for any real x, the set

$$lev_{\alpha}(f) = \{x \in M: f(x) \leq \alpha\} \ldots (8)$$

The level set of f is convex.

2.8 Proposition: (Closeness of the level sets)

If a convex function f is close, then all its level sets are closed.

2.9 Corollary: *The distance function $\|.\|_{\mathbb{R}^N}$ is convex and its level set is closed.*

The convexity of the distance function remains a uniform distribution of data on a symmetric interval.

2.10 Proposition [2]: If $f: I \to \mathbb{R}$ is convex, then either f is monotonic on $Int(I)$, or there exists an $\xi \in Int(I)$ such that f is non increasing on the interval $(-\infty, \xi) \cap I$ and non-decreasing on the interval $[\xi, \infty) \cap I$.

The structure of the intersections $(-\infty, \xi) \cap I$ and $[\xi, \infty) \cap I$ will form a homology on an $n - manifold$.

3- Homology and Geometric on Manifold learning

As we mentioned before, in the field of manifold learning we investigated on the geometric properties of a cloud of random points in $\mathbb{R}^N$. We can rearrange these random points with the notion of "nearst" to each other. This procedure demands an arbitrary parameter (ε) to create simplicial complexes such that it be homotopy equivalent to an $n - dimension\ manifold$.

3.1 Theorem [10]: Let X be unknown subspace of $\mathbb{R}^N$ with finite Lebesgue measure. Let $x_1, \ldots, x_n$ be $n - independent$ random samples uniformly distributed on X. We can find ε for which the union for balls

$$U = \bigcup_{i=1}^{n} B_{\varepsilon_i}(x_i) \ldots (9)$$

Is homotopy equivalent to X.

Now, let us discuss the notion of $B_{\varepsilon_i}(x_i)$.

3.2 Definition [11]: (Cech Complex)

Let $X_n = \{x_1, \ldots\}$ be a collection of points in $\mathbb{R}^N$, and let $\varepsilon > 0$. The *Cech complex* $\tilde{C}(X_n, \varepsilon)$ is constructed as follows:

1- The $0 - complexes$ (vertices) are the points in X_n.
2- An $n - simplex$, $\{x_1, \ldots, x_n\}$ is in $\tilde{C}(X_n, \varepsilon)$ if

$$\cap_{i=0}^{n} B_{\varepsilon_i}(x_i) \neq \emptyset \ldots (10)$$

Where $B_{\varepsilon_i}(x_i)$ stands for the Euclidean ball of radius ε around x_i. In [9], Nerve theorem discussed the links between $\tilde{C}ech\ complex$ and the union $U(X_n, \varepsilon)$ of the covering neighborhood of M and that under specific value of ε.

3.3 Theorem [9]: (Threshold for contractibility, random $\tilde{C}ech\ complex$)

For a uniform distribution on a smoothly bounded convex body K in $\mathbb{R}^N$, there exists a constant C, depending on K, such that if $\varepsilon \geq C\left(\frac{\log n}{n}\right)^{\frac{1}{N}}$ then the random $\tilde{C}ech\ complex\ C(X_n, \varepsilon)$ is a.a.s contractible.

3.4 Definition: Let $A = \{A_1, A_2, \dots, A_k\}$ be a cover of a topological space X. Then the nerve of the cover A is the (abstracted) simplicial complex $\aleph(A)$ on a vertex set $[k] = \{1,2, \dots, k\}$ with $\sigma \subset [k]$ a face whenever $\cap_{i\in\sigma} A_i \neq \emptyset$.

3.5 Theorem [9]: (Nerve Theorem)

Suppose that X is a topological space, and let $U = \{U_\alpha\}_{\alpha\in[k]}$ be any covering of X, and suppose that the covering consists of open sets and is numerable. Suppose further that for all $\emptyset \neq S \subseteq [k]$, we have that $\cap_{s\in S} U_s$ is either contractible or empty. Then $\aleph(U)$ is homotopy equivalent to X.

The parameter quantity ε will control the behavior of such nerve.

3.6 Theorem [3]: Let M be a compact Riemannian manifold. Then there is a positive number e so that $C(M, \varepsilon)$ is homotopy equivalent to M whenever $\varepsilon \leq e$. Moreover, for every $\varepsilon \leq e$ there is a finite subset $V \subseteq M$ so that the sub complex of $C(V, \varepsilon) \subseteq C(M, \varepsilon)$ is also homotopy equivalent to M.

Notation: We say that $G(X_n, \varepsilon)$ asymptotically almost surely (a.a.s) has property P if $\Pr(G(X_n, \varepsilon) \in P) \to 1\ as\ n \to \infty$.

Then, from Theorems [1.3, 3.5 and 3.6] and notation above we conclude the following

3.7 Corollary: *Let $M \subseteq \mathbb{R}^N$ be a compact Riemannian manifold. Set $\{x_1, \dots, x_n\} \subseteq M$ to be the set of random points distributed uniformly on M. Then there exists a constant C depending on M such that $n\varepsilon^N \geq C \log n$. Then*

$$lim_{\,n\to\infty}\, Pr\left\{\left(U_{i=1}^{n}\left(B_{\varepsilon_i}(x_i), \varepsilon_i\right) \cong M\right\} = 1 \dots (11)$$

The homology of M is a collection of abelian groups $\{H_k M)\}_{k=0}^{\infty}$. The zeroth homology $H_0(M)$ represents the elements of M which generate the connected components of M. Such elements can be thought of as a geometric graph.

3.8 Definition [9]: Let $f: \mathbb{R}^N \to \mathbb{R}$ be a probability density function, let $x_1, \dots, x_n$ be a sequence of independent and identically distributed $N - dimensional$ random variables with common density f, and let $X_n = \{x_1, \dots, x_n\}$. The geometric graph $G(X_n, \varepsilon)$ is the geometric graph with vertices X_n, and edges between every pair of vertices u, v with $d(u, v) \leq \varepsilon$.

AFAK for Sciences Journal مجلة آفاق للعلوم EISSN: 2602-5345 ISSN: 2507-7228

Moreover,

3.9 Proposition [13]: Let X_n be any finite collection of points $x_1, \dots, x_n \in \mathbb{R}^N$ such that it is $\frac{\varepsilon}{2}$ dense in M, i.e., for every $p \in M$ there exists an $x_i \in X_n$ such that $\|p - x_i\|_{\mathbb{R}^N} < \frac{\varepsilon}{2}$. Then for any $\varepsilon < \sqrt{\frac{3}{5}}\tau$, we have that U deformation retracts to M. There for homology of U equal homology of M, where $U = \cup_{x_i \in X_n} B_\varepsilon\ (x_i)$, and $B_\varepsilon(x_i)$ represent an open ball of radius ε around $x_i \in X_n$.

$$\tau = inf_{p \in M}\sigma(p) \equiv the\ distance\ of\ p\ to\ the\ medial\ axis$$

More specifically and for the cardinality of X_n (the number of connected components).

3.10 Theorem [18]: Let X_n be drawn by sampling M in $i.i.d$ fashion according to the uniform probability measure on M. Then with probability $1 - \delta$, we have that X_n is $\frac{\varepsilon}{2} - dense$ $\left(\varepsilon < \frac{\tau}{2}\right)$ in M provided

$$|X_n| \geq \beta(\varepsilon)\left(\ln \beta\left(\frac{\varepsilon}{2}\right) + \ln\left(\frac{1}{\delta}\right)\right) \dots (11)$$

Where, $\beta(x) = \dfrac{Vol(M)}{2^{\frac{(cos\theta)^k}{k+1}} I_{\frac{1-x^2cos^2\theta}{16\delta^2}}\left(\frac{k+1}{2}, \frac{1}{2}\right) Vol(B_x^k)}$

From Proposition (3.9) we can treat the $\varepsilon - density$ in M with the notion of distance function to create a $\aleph(U)$. Bobrowski [11] take this problem as considerable situation.

4- Distance Functions, Excursion Sets and Homology of Manifold

Let a collection of points $x_1, \dots, x_n \in \mathbb{R}^N$ be $\varepsilon - dense$ in M. We can define the distance function $d_\varepsilon: \mathbb{R}^N \to \mathbb{R}$ on it as follows:

$$d_\varepsilon \coloneqq inf_{p \in M}\|x_i - p\|_{\mathbb{R}^N}\ \forall p \in M, x_i \in \{x_1, \dots, x_n\} \dots (12)$$

So, we can set $B_\varepsilon(x_i) \coloneqq \{p \in M: d_\varepsilon(x_i, p) \leq \varepsilon\}$. In [11] Bobrowski, Mukherjee had defined the sub level of such distance function. And according to propositions (2.7, 2.8 and 2.10) we can set

$$d_\varepsilon^{-1}\big((-\infty, \varepsilon]\big) \coloneqq \{p \in M: d_\varepsilon(x_i, p) \leq \varepsilon\} \dots (13)$$

Moreover, the same author Bobrowski in [10] had given more specification on such sub level set as:

$$d_\varepsilon^{-1}\big((-\infty, \varepsilon]\big) = U_\varepsilon \approx \tilde{C}(X_n, \varepsilon) \dots (14)$$

Which is taken as random field contains $X_n = \{x_1, \dots, x_n\}$ as $i.i.d$ random points which is distributed on M to create an equivalent homotopy to M. These sublevel sets generate a filtration to the cloud of point in M.

4.1 Definition [3]: Given a point cloud X_n, a reference map $\rho: X_n \to Z$ to a metric space, and covering $U = \{U_{x_i}\}_{x_i \in X_n}$, we define a simplicial complex $SS: SS(X_n, \rho, U)$ as follows: The vertices of SS are pairs (x_i, I), where $x_i \in X_n$ and where I is stability interval for the point cloud $X_n = \rho^{-1}(U_{x_i})$. A $(k+1) - tuples$ $\{(x_0, I_0), (x_1, I_1), \dots, (x_n, I_n)\}$ spans a $k - simplex$ in SS if:

1- $U_{x_0} \cap \dots \cap U_{x_k} \neq \emptyset$.

2- $I_0 \cap \dots \cap I_k \neq \emptyset$.

Then, the vertex map $[x, I] \to x$ induces a map of simplicial complex $\rho: SS \to \tilde{C}(U)$. By scale choice for X_n and U, we will mean a section of the map ρ, i.e., a simplicial map $s: \tilde{C}(U) \to SS$ such that $\rho \circ s = Id_{\tilde{C}(U)}$.

Moreover, from the sublevel set $d_\varepsilon^{-1}((-\infty, \varepsilon])$ we can extract an $\mathbb{N} - persistence$ homology.

4.2 Proposition [3]: Any tame $\mathbb{N} - persistence\ F - vector$ space $\{V_n\}_n$ can be decomposed as:

$$\{V_n\}_n \cong \bigoplus_{i=0}^{\mathbb{N}} U(m_i, n_i) \dots (15)$$

Where, each m_i is a non – negative integer, and n_i is a non – negative integer or $+\infty$. The decomposition is unique in the sense that the collection of pairs $\{(m_i, n_i)\}_i$ is unique up to an ordering of factors.

4.3 Corollary: *Let Mbe a compact sub manifold of $\mathbb{R}^N$. Let $X_n = \{x_1, \dots, x_n\}$ be a set of points drawn in $i.i.d$ fashion according to the uniform probability measure on M. Let $\varepsilon > 0$ and set n_i above as $(+\infty)$. Then the excursion set $d_\varepsilon^{-1}([\varepsilon, +\infty))$*

$$\{V_{\varepsilon_n}\}_n \cong \bigoplus_{i=1}^{\mathbb{N}} U(\varepsilon_i, +\infty) \dots (16)$$

or

$$\{V_{\varepsilon_n}\}_n \cong \bigoplus_{i=1}^{\mathbb{N}} d^{-1}(\varepsilon_i, +\infty) \dots (17)$$

Form an $\mathbb{N} - persistence\ F - vector\ space$ $\{V_{\varepsilon_n}\}_n \equiv \{B_{\varepsilon_i}\}_{i=1}^{n}$ for M.

Notation: The excursion sets are symmetric, so we observe that $d_\varepsilon^{-1}([\varepsilon, +\infty)) \approx d_\varepsilon^{-1}((-\infty, \varepsilon])$.

The geometric quantities relate to the excursion set $d^{-1}([\varepsilon, +\infty))$ can be extract upon it is curvature. Taylor [4,5] had discussed the ability of Lipschitz – Killing curvature to do so.

4.4 Theorem [4]: Let X_n be a suitably regular, centered, unite variance Guassian field on a C^3 compact manifold M. Then:

$$E\left[\chi\left(M \cap f^{-1}[\varepsilon, +\infty)\right)\right] = \sum_{i=0}^{n} \mathcal{L}_i(M)\rho_i(u) \dots (18)$$

Where $\mathcal{L}_i(M)$ are the Lipchitz – Killing curvature of M, calculated with respect to the metric induced by f and ρ_i is given by:

$$\rho_i(u) = \frac{1}{2\pi^{\frac{(j+1)}{2}}}\int_u^\infty H_j(t)e^{-\frac{t^2}{2}}\,dt = \begin{cases} \frac{1}{2\pi^{\frac{j+1}{2}}} H_{j-1}(u)e^{-\frac{t^2}{2}} & ; j \geq 1 \\ 1-\varphi(u) & ; j = 0 \end{cases} \dots (19)$$

Where, $H_j(x) = \sum_{i=0}^{\frac{j}{2}} \frac{(-1)^i i!}{(j-2i)!\,i!\,2^i} x^{j-2i}$ is the Hermit polynomial and φ is the cumulative distribution of a standard Gaussian random variable.

For more details about Lipchitz – Killing curvature the reader should return to Adler [14] on his work to extract intrinsic volume of the tubes $Tube(A, \varepsilon) = \{x \in \mathbb{R}^N; d(x, A) \leq \varepsilon\}$. The same author in [16] had discussed the ability of counting the number of connected components upon excursion sets of random $i.i.d$ Gaussian field. On the area of algebraic topology we know that the Betti numbers is a topological invariant measuring the number of connected components and holes of different dimension. The notion of Euler Characteristic function (EC) , do the same work. By the other hand, Morse's theorem relates (EC) to the critical points of a nice function.

5- Critical Points, Excursion Sets, and Homology on Manifolds:

We will discuss the behavior of the critical points of the distance function as $i.i.d$ Gaussian random field. Upon Morse's theorem this points counts the number of connected component of $n - dimension\, M$, and according to Definition (4.1) Theorem (4.2) and Corollary (4.3) it creates an $\mathbb{N} - persistence$ homology on M.

5.1 Definition [10]: For a finite set X_n of points in $\mathbb{R}^N$, of size $|X_n|$. Let $d_{X_n}: \mathbb{R}^N \to \mathbb{R}$ be the distance function for X_n, so that:

$$d_{X_n}(x) = min_{x \in X_n} \|x - p\|_{\mathbb{R}^N}, p \in \mathbb{R}^N \dots (20)$$

5.2 Definition [10]: A point $c \in \mathbb{R}^N$ is a critical point of d_{X_n} with index $1 \leq k \leq$ N if there exists a subset Y of $k + 1$ points in X_n such that:

1- $\forall y \in Y: d_{X_n}(c) = \|c - y\|_{\mathbb{R}^N}$ and for all $p \in X_n/Y$ we have $\|c - p\|_{\mathbb{R}^N} > d_{X_n}(p)$.

2- The points in Y are in generated position (i.e., the $(k+1)$ points of Y do not lie in a $(k-1)-dimensional$ affine space.
3- $c \in conv^0(Y)$, where $conv^0(Y)$ is the interior of the convex hull of Y (an open $k - simplex$ in this case).

Euler characteristic is a topological invariant describes the shape of the space:

$$\chi(x) = \sum_{k=0}^{N} (-1)^k \beta_k \ldots (21)$$

Where X is an $n - dimensional$ compact and β_k is the k – th Betti number. Morse's theorem represents the critical values of a nice function as topological invariant to do the same as Betti numbers.

5.3 Theorem [16]: The functional $(\varphi \equiv Euler\ characteristic)$ has the following definition for a nice set $A \in \mathbb{R}^N$.

$$\varphi(A) = \begin{cases} number\ of\ disjoint\ closed\ interval\ in\ A & if\ N = 1 \\ \sum \{\varphi(A \cap \varepsilon_x) - \varphi(A \cap \varepsilon_{x^-})\} & if\ N > 1 \end{cases} \ldots (22)$$

And where ε_x denote the $(N-1)$ plane of points in $\mathbb{R}^N$ all of which have their j – th coordinate equal to x, $j \in [1, N]$.

5.4 Theorem [16]: (Morse's theorem) Let $f(x), x \in \mathbb{R}^N$ be a real – valued function of class C^2, admissible relative to a compact C^2 domain $A \subset \mathbb{R}^N$ with C^2 boundary and a finite number of components. Then the Euler Characteristic of A is given by:

$$\varphi(A) = \sum_{k=0}^{N} (-1)^k m_k + \sum_{k0}^{N-1} (-1)^k m'_k \ldots (23)$$

Where, $\{m_k\}_{k=0,..,N}$ and $\{m'_k\}_{k=0,\ldots,(N-1)}$ is the number of critical point of $f_{\chi(A)}$ and $f_{\chi(\partial A)}$ respectively.

Now, from Definition (5.2) we can set $f(x)$ above as d_{X_n} and take it's critical values as $i.i.d$ randomly on M. Now, and for the limiting behavior of the set of critical points and from Theorem (3.1) we can deal with the term $\varepsilon \geq C\left(\frac{\log n}{n}\right)^{\frac{1}{N}}$ since it control the persistence homology of M. So, we can set:

$$n\varepsilon^N \geq C \log n \ldots (24)$$

Now, let:

$$C_{n,k}^L \coloneqq \left\{x \in C_{n,k} : d_{B_{\varepsilon_n}}(x) \leq \varepsilon_n\right\} = \left\{C_{n,k} \cap U\left(B_{\varepsilon_{n,k}}(x), \varepsilon_n\right)\right\} \ldots (25)$$

AFAK for Sciences Journal مجلة آفاق للعلوم EISSN: 2602-5345 ISSN: 2507-7228

Where, $C_{n,k}^{L}$ stands for local critical points in M, and let $\widehat{N}_{n,k} = \left|C_{n,k}^{L}\right|$ as in Definition (1.2).

5.5 Definition [11]: (Critical point of the distance function)

If $n\varepsilon^{N} \geq C \log n$, then:

1- If $C > (\omega_n f_{min})^{-1}$, then

$$\lim_{n\to\infty} P\left(N_{n,k} = \widehat{N}_{n,k} \forall 1 \leq k \leq N\right) = 1 \ldots (26)$$

2- If $C > 2(\omega_n f_{min})^{-1}$, then almost surely there exists $m > 0$ (possiply random), such that for $n > m$

$$N_{n,k} = \widehat{N}_{n,k} \forall 1 \leq k \leq N \ldots (27)$$

Where, $\widehat{N}_{n,k}$ stands for the number critical points of the distance function, and $N_{n,k}$ is the number of critical points of $U\left(B_{\varepsilon_n}, \varepsilon_n\right)$.

By the other hand, the critical points of the distance function represent a Gaussian random field according to its limit behavior.

5.6 Theorem [10]: If $nr_n^N \to \infty$, and f is lower bounded with convex support, then for $1 \leq k \leq N$

$$\lim_{n\to\infty} n^{1} E\left\{N_{n,k}^{G}\right\} = \gamma_k(\infty) \ldots (28)$$

$$\lim_{n\to\infty} n^{-1} var\left(N_{n,k}^{G}\right) = \sigma_k^2(\infty) \ldots (29)$$

And

$$\frac{N_{n,k}^{G} - E\left\{N_{n,k}^{G}\right\}}{\sqrt{n}} \xrightarrow{\mathcal{L}} \mathrm{N}\left(0, \sigma_k^2(\infty)\right) \ldots (30)$$

5.7 Proposition [10]: Under the condition of the forgoing theorem, and if $nr_n^N \geq D^* \log n$, for sufficiently large $(f - dependent)D^*$, then for $1 \leq k \leq N$:

$$\lim_{n\to\infty} E\left\{N_{n,k}^{G} - N_{n,k}\right\} = 0 \ldots (31)$$

Where $N_{n,k}^{G}$ stands for global number of critical points of the distance function. If we replace $|X_n|$ with the quantity $N_{n,k}^{G}$ we can set from Theorem (3.10) that

5.8 Corollary: *Let $N_{n,k}^{G}$ be the number of critical points of the distance function as in Theorem (5.6) and Proposition (5.7) above. Set $d_{X_n} \coloneqq inf_{x\in M}\|x\|_{\mathbb{R}^N}$, which distributed randomly on an $n - dimensional\ manifold\ M$ according to uniform probability measure on M, then:*

$$Pr\left(H_k\left(U_{\varepsilon_n}\left(C_{n,k}^{G}, \varepsilon_n\right)\right) \cong H_k(M)\right) \geq \frac{2^k C\, Vol(M)}{\omega_1 \varepsilon_n^k} e^{-\frac{N_{n,k}^{G}\omega_1\varepsilon_n^k}{Vol(M)}} \ldots (32)$$

By the way, since we deal with $i.i.d$ of random Gaussian field of critical points of distance function, then we need to use a density function $D: M \to \mathbb{R}^N$ such that $D \circ d_{\varepsilon_n}^{-1}([\varepsilon_n, +\infty)) = Id_{C_{n,k}^G}$ as in Definition (4.1)

5.9 Definition [16]: Let $F: \mathbb{R}^N \to \mathbb{R}$ be C^2 over a compact $T \subset \mathbb{R}^N$. Then if the X_k are all finite, the $DT(differential\ topology)$ characteristic of the excursion set $A_n(F,T)$ is given by:

$$\chi(A) = (-1)^{N-1} \sum_{k=0}^{N-1} (-1)^k \chi_k \dots (33)$$

Where χ_k is the number of points $t \in T$ satisfying the following conditions:

$$F(t) = u$$

$$F_j(t) = 0 \quad j = 1, \dots, N-1$$

$$F_N(t) > 0$$

The index of $D(t)$ equals k. Where $D(t)$ is the $(N-1) \times (N-1)$ matrix of second derivative with elements $X_{ij}(t), i,j = 1, \dots, (N-1)$.

More specifically

5.10 Theorem [16]: (For excursion sets) Let $F: \mathbb{R}^N \to \mathbb{R}$ be C^2 over a compact $T \subset \mathbb{R}^N$and assume that $F(t) < \lambda$ for all $t \in \partial T$. Then, if the χ_k defined above, are all finite. The Euler characteristic of excursion set $A_u(F,T)$ is given by:

$$\varphi(A) = (-1)^{N-1} \sum_{k=0}^{N-1} (-1)^k \chi_k \dots (34)$$

5.11 Theorem [16]: Let $X(t)$ be zero mean, homogeneous Gaussian random field on $\mathbb{R}^N$ and let $T \subseteq \mathbb{R}^N$ be a compact set. Suppose that X has almost surely continuous partial derivatives of up to second order with finite variance, that the joint distribution of X and these partial derivative is non degenerate and that the moduli of continuity of the X_{ij} satisfy:

$$P\{max_{i,j} \omega_{ij}(h) > \varepsilon\} = o(h^N) \ \ as\ h \downarrow 0 \dots (35)$$

Notation: For non – negative functions $g\ and\ h$ we say $g(n) = o(h(n))$ if and only if $\exists n_0\ and\ k; for\ n > n_0$ we have $g(n) \leq k.h(n)$.

From the basic large deviation theorem.

5.12 Theorem [13]: (Basic Large Deviation) Let X be centered Gaussian random variable with variance σ^2. Then, we have for all $\varepsilon > 0$:

$$\left(1-\frac{\sigma^2}{\varepsilon^2}\right)\frac{\sigma}{\varepsilon\sqrt{2\pi}}\exp\left(-\frac{\varepsilon^2}{2\sigma^2}\right)P\{X>\varepsilon\}\frac{\sigma}{\varepsilon\sqrt{2\pi}}\exp\left(-\frac{\varepsilon^2}{2\sigma^2}\right)\dots(36)$$

This implies that:

$$\lim_{\varepsilon\to 0}\varepsilon^{-2}\log P\{X>\varepsilon\} = -(2\sigma^2)^{-1}\dots(37)$$

5.13 Corollary: *Let $X_n \in M \subseteq \mathbb{R}^N$ be zero mean, homogeneous Gaussian field on $\mathbb{R}^N$ with finite variance. Set $X_n = \{x_1,\dots,x_n\}$ as critical points of the distance function $d_\varepsilon: M \to \mathbb{R}^N$ distributed according to $Id_{C^G_{n,k}}$. Then $\exists n_0$ and ksuch that for $n > n_0, n\uparrow\infty, \varepsilon_n \downarrow 0$:*

$$Pr\{x_i \in M: d_{\varepsilon_n}(x_i, C^G_{n,k}) > \varepsilon_n\} \le kh(\varepsilon_n)\dots(38)$$

Moreover,

$$\lim_{\varepsilon_n\to 0}\varepsilon_n^{-2}\log P\{x_i\in X_n: d_{\varepsilon_n}(x_i,p)>\varepsilon_n, p\in M\} = -(2\sigma^2)^{-1}\dots(39)$$

In [14] Adler had considered the situation $E[\chi(D\cap f^{-1}[u,+\infty))]$, this statement represents the geometric properties of the parameter space D, where $f^{-1}[u,+\infty)$ is the sublevel set of f. Since

$$A_\varepsilon(d_{\varepsilon_n}, X_n) = \{x_i\in X_n: (d_{\varepsilon_n}\circ G)(x_i)\ge\varepsilon_n\}$$
$$= \{x_i\in X_n: G(x_i)\in d^{-1}_{\varepsilon_n}[\varepsilon_n,+\infty)\}$$
$$= M\cap d^{-1}_{\varepsilon_n}[\varepsilon_n,+\infty)\dots(40)$$

Where A_ε as in Definition (5.9), and G stands for the Gaussian distribution function on $X_n \subset M$. By the other hand in [16] we have that:

$$E\{\chi(A)\} = |T|\rho_N(\lambda)\dots(41)$$

Where A stands here for excursion set and $T\subset X\subseteq\mathbb{R}^N$ is the set of $i.i.d$ random Gaussian field on X.

Now, from Theorems (4.4, 5.12, 5.14) and Equations (39, 40, 41) and definition of the conditional term τ in Proposition (3.9), we had

5.14 Corollary: *Let $M\subseteq\mathbb{R}^N$ be a compact Riemannian manifold, set $d_{\varepsilon_n}:\mathbb{R}^N\to\mathbb{R}$ as a distance function. Moreover, let $C^G_{n,k}(1\le k\le N)$ be the set of the critical points of the distance function d_{ε_n}, with $N^G_{n,k} = |C^G_{n,k}|$. Let $\varepsilon_n \ge C\left(\frac{\log n}{n}\right)^{\frac{1}{N}}$, then:*

$$E\left\{\chi\left(M \cap d_{\varepsilon_n}^{-1}[\varepsilon_n, +\infty)\right)\right\} = E\left\{\chi\left(M \cap U_{\varepsilon_n}\left(C_{n,k}^G, \varepsilon_n\right)\right)\right\} = N_{n,k}^G \rho_N(\varepsilon_n) \dots (42)$$

Where,

$$\rho_N(\varepsilon_n) \coloneqq \frac{exp\left(-\frac{\varepsilon_n^2}{2\tau^2}\right)(det\Lambda)^{\frac{1}{2}}}{(2\pi)^{\frac{N+1}{2}}\tau^N} H_{N-1}\left(\frac{\varepsilon_n}{\tau}\right) \approx \frac{C}{\tau^N (2\pi)^{\frac{N+1}{2}}} e^{-\frac{\varepsilon^2}{2\tau^2}} \dots (43)$$

And $\varepsilon_n \coloneqq inf_{p \in M} \|p - c_i\|_{\mathbb{R}^N}, 1 \le i \le n \ \forall c_i \in C_{n,k}^G$.

6- Conclusion:

The ***internal geometric structures*** of an $n - dimensional\ manifold$ take a big realm in many areas of search, such as biology, chemistry, engineering, etc. Such problem had a wide space of work. The technique of ***clustering or partitioning*** such spaces to ***groups and components*** can be used to ***decrease the loosing information*** to guarantee the best ***optimization geometric structure***. This technique demands careful choosing of ***maps and parameters*** to build an ***optimize structure***.

Reference:

1- Atsushi Kasue; A Compactification of Manifold with Asymptotically Nonnegative Curvature; Annals Scientific, Vol. 21, No. 4, (1988), pp. 593 – 622.
2- Constantine P. Niculescu, Lars – Erik Persson; Convex Functions and Their Applications; A contemporary Approach; Monograph – September 16, 2014, Springer.
3- Gunnar Carlsson; Topology and Data; American Mathematical Society; Vol. 46, No. 2, April 2009, pp. 255 – 308.
4- Jonathan E. Taylor, Robert J. Adler; Euler Characteristic for Gaussian Fields on Manifolds; The Annals Probability, Vol. 31, No. 2, pp. 533 – 563, 2003.
5- Jonathan E. Taylor; A Gaussian Kinematic Formula; The Annals of Probability; Vol. 34, No. 1, pp. 122 – 158, 2006; Doll:10.1214/009117905000000594.
6- Karol Borsuk; On the Embedding of Systems of Compacta in Simplicial Complexes; Fundamental Mathematica; Vol. 35, No. 1, (1948), pp. 217 – 234, ISSN: 0016 – 2736.
7- Mikhail. Gromove; Metric Structure for Riemannian and Non – Riemannian Spaces; e Book (2007), Springer, ISBN 978 – 0 – 8176 – 4583 – 0.
8- Matthew Kahle, Elizabeth Meckes; Limit Theorem for Betti Numbers of Random Simplicial Complexes; arXiv:1009.4130v3[math.PR]18Jane 2011.
9- Matthew Kahle; Random Geometric Complexes; arXiv: 0910.1649v3 [math.PR] 6 Dec 2010.
10- Omer Bobrowski; Algebraic Topology of Random Fields and Complexes; Research Thesis. Submitted to the senate of the Technion - Israel Institute of Technology. July 2012. Haifa.
11- Omer Bobrowski, Sayan Mukherjee; The Topology of Probability Distributions on Manifolds; arXiv: 1307.1123v2 [math.PR] 1Mar 2014.
12- Omer Bobrowski, Robert J. Adler; Distance Functions, Critical Points and the Topology of Random Cech Complexes; arXiv: 1107.4775v2 [math.PR] 11 Aug 2014.
13- P. Niyogi, S. Smale, S. Weinberger; Finding the Homology of Sub Manifolds with High Confidence From Random Samples; Discrete & Computational Geometry, Vol. 39, pp. 419 – 441 (2008). Springer Link.
14- Robert J. Adler, Jonathan E. Taylor, Keith J. Worsley; Applications of Random Fields and Geometry; e Book. Springer Monograph in Mathematics. 2007. Doll: 10.1007/978 – 0 – 387 – 48116 – 6.
15- Robert J. Adler, Omer Bobrowski, Matthew S. Borwan, Eliran Subag, Shmuel Weinberger; Persistent Homology for Random Fields and Complexes; IMS Collections. Theory Powering Applications, Vol. 6 (2010), pp. 124 – 143; Doll: 10.1214/10 – IMS COLL 609.
16- Robert J. Adler; On Excursion Sets, Tube Formulas and Maxima of Random Fields; The Annals of Applied Probability, Vol. 10, No. 1, pp. 1 – 74, 2000.
17- R. Lowen, M. Sioen; Approximations in Functional Analysis; Result. Math. 37 (2000), pp. 345 – 372; Birkhauser Verlag, Basel, 2000.
18- Yang Wang, Bei Wang; Topological Inference of Manifolds with Boundary; arXiv: 1810.05759v1 [stat. Co] 12 Oct 2018.