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Diameter of Convex Sets (Bodies) & Concentration of Measure

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Abstract :

Under the notion of concentration of measure and (LDP) we found that the diameter of a section of any convex bodies will depends basically in the isotropic constant L_K of the polar body of the convex body and we found that

$$diam(K \cap E) \leq C\sqrt{p}diam(K \cap K^0)$$

where $E \subset K^0$.

Key words: Embedding, Concentration of Measure, Large Deviation Principle, Polar Body

الملخص:

تحت مفهوم تركيز الحجم و مبدأ الانحراف الأعظم

وجدنا أن قطر القطاع لأي جسم محدب يعتمد بصورة

أساسية على ثابت التحدب $\square_K$ للجسم القطبي لأي

جسم محدب. و وجدنا أنه

$$\boldsymbol{diam(K \cap E) \leq C\sqrt{p}diam(K \cap K^0)}$$

. $\boldsymbol{E \subset K^0}$

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1. INTRODUCTION

The isometric embedding is a mapping $f: X \to Y$ where X is a metric space with a metric ρ and Y is a metric with a metric σ. In embedding we want to find a plausible relation $(\rho \sim \sigma)$ in such way that $\sigma\big(f(X), f(Y)\big) \propto \rho(X, Y)$; for example the legal norm $\|.\|$ for a normed space and canonical Euclidean norm $|.|$. We say that a mapping $f: X \to Y$ whose X and Y as above is $D - Embedding$ where $D \geq 1$ is a real number, if there exists a number $r > 0$ such that for all $x, y \in X$ the relation

$$\big\{\rho(x, y) \leq \sigma\big(f(x), f(y)\big) \leq D.r\rho(x, y)\big\} \ldots\ldots (1)$$

exist.

With the forgoing definition, the notion of isometric embedding will relevant in some way with the concentration of measure phenomenon in such way that the Equation (1) above can be seen as

$$P\left(\sigma\big(f(x), f(y)\big) \leq C\rho(x, y)\right) \leq C_1 e^{-\frac{C^2[\rho(x,y)]^2}{\tau^2}} \ldots\ldots (2)$$

Where C is universal constant and $C_1 = C_1(n, \tau)$

The quantity τ plays an important role in the process which is required as covariance of the point around the neighborhood.

Kashin was discovered that there is a subspace of $\mathbb{R}^N$ of dimension proportional to N on which the ℓ and ℓ_2 are equivalent. In the same way an inner product, which is a symmetric bilinear positive definite mapping $\langle .,. \rangle: V \times V \to \mathbb{R}$ on a real vector space, assists in the construction of geometry. An orthonormal set basis of X is an orthonormal set $\{\varphi_i\}_{i \in [1,k]}$such that all finite linear combinations $\sum_{i=1}^{k} \varphi_i x_i$ is dense in X. This complexity aids much better in the construction of the spaces.

Our paper concerns the diameter of these subspaces as sections of convex bodies. We investigate Large Deviation Principle (LDP) and its connection to the concentration of measure phenomenon. These mathematician process aids powerfully in the measurement of the diameter of the sections of convex bodies. Dvrotzky – Rogers in their Lemma had treated the problem of the length of the orthogonal coordinate upon concentration on a unite Euclidean ball.

Our paper is organized as follows. In the 1st section we discuss how embedding, which is a powerful tool in concentration of measure, affects to describe the geometrical property of any body. The 2nd section describes the notion of (LDP) and concentration of measure which is a powerful technique to describe and built a section of anybody in the space. Our 3rd section investigates the convex bodies and its geometrical properties upon embedding, also we describe the notion of the polar body K^0 of any convex body and use it as concentration tool to describe the sections of any convex body. At last we had a short discussion.

Embedding and Construction of geometry

Embedding plays an important role in the construction geometry of spaces. If we embed to a random $k-dimensional$ subspace, such subspace can be chosen by selecting an orthonormal basis $(\varphi_1, \varphi_2, \ldots, \varphi_k)$ which is random $k-tuple$ of unite orthogonal vector$\{\|\varphi_i\| = 1\}_{i=1}^{k}$, where the coordinate of the projection from X to Ψ can be produced in the manner of scalar product $\langle x, \varphi_1\rangle, \ldots, \langle x, \varphi_k\rangle$, by the other hand the orthonormal basis $\varphi_1, \ldots \varphi_k \in \Psi$ is chosen identically and independent where $X \longmapsto \langle x, \varphi_i\rangle \; (1 \le i \le k)$. Dvrotezky and Rogers state in their Lemma that the construction geometry of random points will be stated with the notion of the scalar product.

Lemma (Dvortzky – Rogers): Let $\|.\|$ be a norm on $\mathbb{R}^n$ with unite ball $\mathcal{B}$, let $\mathcal{E} \subseteq \mathcal{B}$ be the John ellipsoid. Then $\exists x_1, \ldots, x_n \in \mathbb{R}^n$ which are orthonormal with respect to $\langle .,. \rangle_{\mathcal{E}}$, and such that

$$\|x_i\| \ge 2^{-\frac{N}{N-i+1}} \quad , i = 1, \ldots, N-1 \ldots \ldots (3)$$

Dvortzky – Rogers Lemma state that, every unite ball contained a constructed geometry upon some orthogonal set of points which built a net on $\mathcal{B}$.

Lemma: Let $\varepsilon \in (0,1)$ and let $\mathcal{M}$ be an $\varepsilon - net$ of S^{n-1}. Then, for all $u \in S^{n-1}$, there a sequences $(u_k)_{k\ge0} \subseteq \mathcal{M}$ and $(\varepsilon_k)_{k\ge1} \subseteq \mathbb{R}$ such that

$$u = u_0 \sum_{k=1}^{\infty} \varepsilon_k u_k$$

$$0 \le \varepsilon_k \le \varepsilon^k, \forall k \ge 1$$

with this scalar product we can connect the norm $\|.\|$ on normed space $(X, \|.\|)$ with the legal Euclidean norm $|.|$ on $\mathbb{R}^n$.

The quantity ε_k controls k (an ambient dimension of the embed space $\big(k = k(\varepsilon, n)\big)$.

Lemma: Let $u_1, \ldots, u_t$ be orthogonal transformations of $\mathbb{R}^n$. Let $\|.\|$ be a norm on $\mathbb{R}^n$ and let $|.|$ denote the canonical Euclidean norm. Put $|\|x\|| = \frac{1}{t}\sum_{i=1}^{t}\|u_i x\|$ and assume $|\|x\|| \le C|x|$ for all $x \in \mathbb{R}^n$ and some $0 < C < \infty$. Then

$$\|x\| \le C\sqrt{t}|x| \ldots \ldots (4)$$

With the assumption of the Ex – Lemma we can conclude with $|\|x\|| \le C|x|$ that $c|x| \le \|x\|_X \le C\|x\|$ with probability greater than $1 - e^{\frac{C^2 t}{\tau^2}}$, which enough with the constraints of the embedding process. Also we can deal with the notion of concentration of measure

Theorem (Jonson – Linden Strauss Flatting Lemma): Let X be an $n - point$ set in a Euclidean space $(i.e\, X \subset \ell_2^n)$ and let $\varepsilon \in (0,1]$ be given. Then there exists a $(1+\varepsilon) - embedding$ of X into ℓ_2^d, where $d = O(\varepsilon^2 \log n)$.

Corollary : *With the assumption of Lemma (1.3). Then*

$$c|x| \le \|x\| \le C|x|$$

with,

$$P\left(\|x\| \leq C\sqrt{t}|x|\right) \leq C_1 e^{-\frac{C^2 t}{\tau^2}} \dots \dots (5)$$

Where, C_1 as we mentioned before.

The basic research of the approximate embedding is all about the quantities (ε, d), and as in the concentration of measure; d is the ambient dimension of the space and ε is the diameter of the open neighborhood.

Theorem: There is a constant $C > 0$ such that for all $\varepsilon > 0$, every $n - dimensional$ normed space admits a subspace whose Banach – Mazure distance from ℓ_2^k is at most $(1 + \varepsilon)$ and $k > \frac{C\varepsilon}{\left(\log\left(\frac{1}{\varepsilon}\right)\right)^2} \log n$.

The relevant of the source dimension and the subspace dimension is created in the following theorem.

Theorem: For some universal constant C

$$\frac{n}{C} \leq kt \leq Cn$$

Where t stand for the number of orthogonal transformation as in Lemma (1.3).

The Banach – Mazure distance between two spaces K and T is $d_{BM}(K, T) = \inf\left\{d_G\left(K, L(T)\right), L\ is\ a\ linear\ operatore\right\}$, where d_G stands for the geometric distance $d_G(K, T) = \inf\left\{a, b; \frac{1}{a}K \subset T \subset bK; a, b > 0\right\}$

With the above theorem we guarantee that $d_{BM} < C_1 \log n$; in a such way that: $c \log n\ (1 - \varepsilon)K \subset T \subset C \log n\, (1 + \varepsilon)K$ where C, c are universal constants. In the same area we had

Theorem: There exist $C > 0$ such that $\forall n \in \mathbb{N}$, $\forall \varepsilon > 0$, every $n - dimensional$ normed linear space X admits a subspace $Y \subseteq X$ such that $d(Y, \ell_2^k) \leq 1 + \varepsilon$, and $k > \frac{C\varepsilon}{\log\left(\frac{1}{\varepsilon}\right)^2} \log n$.

Theorem (Milman's Dvortzky Theorem): There exist a function $C(\varepsilon)$ such that, for all $k \leq c\varepsilon^2 E(X^2)$, $\ell_2^k \hookrightarrow^{(1+\varepsilon)} X$. Actually one may take $k \leq c\varepsilon^2 E(X^2)$.

Also classical Dvortzky theorem takes the same way

Theorem (Classical Dvortzky): Let X be a normed space of dimension n. Then there exist a function $C(\varepsilon) \geq 0$ such that, for all $k \leq C(\varepsilon) \log n$, $\ell_2^k \hookrightarrow^{(1+\varepsilon)} X$.

Which mean that $\|x\| \leq C\sqrt{k}|x|$, where $k \approx C(\varepsilon) \log n$. So $P\left(\|x\| \leq C\sqrt{k}|x|\right) \leq C_1 e^{-\frac{kC^2|x|^2}{\sigma^2}}$

By the other hand the notion of $Lipschitz - function$ play a basic role in the area of concentration of measure.

Lemma (Lipschitz extension): Given a function f defined on a finite subset $x_1, \dots, x_n$ of X, there exist a function f' which coincides with f on $x_1, \dots, x_n$ is defined in the whole space X, and has the same Lipschitz constant as f. Additionally, it is possible to explicitly constructed f' in the form:

$$\boldsymbol{f'} = \boldsymbol{\alpha \min_{i=1,\dots,n}\big(f(x_i) + L(f)d(x, x_i)\big) + (1-\alpha)\max_{i=1,\dots,n}\big(f(x_i) - L(f)d(x, x_i)\big)} \;\dots\dots\; \mathbf{(6)}$$

for any $\alpha \in [0,1]$, with $L(f) = \max_{i,j=1,\dots,n} \frac{\left(f(x_i)-f(x_j)\right)}{d(x_i,x_j)}$. □

So as a consequence $\frac{|f(x_i)-f(x_j)|}{d(x_i,x_j)} \leq L(f)$ so $|f(x_i) - f(x_j)| \leq L(f)d(x_i, x_j)$ and :

$$P\left(\left|f(x_i) - f(x_j)\right| \leq L(f)d(x_i, x_j)\right) \leq \left(2e^{-\frac{(L(f))^2}{\tau^2}} \approx 2\alpha(X;\varepsilon)\right) \dots\dots (7)$$

Notation: The Formula (6) above is equivalence with the definition of the convex body. So the $Lipschitz - function$ could create a convex body in $\mathbb{R}^n$.

Every bodies in $\mathbb{R}^n$ with simple assumption could create a convex body of it in $\mathbb{R}^n$ using concentration of measure phenomenon, in such way that their norms are equivalent.

Theorem: Let $K \subset \mathbb{R}^n$ be a body with $L_K < A$, and let $0 < \lambda < 1$. Then for any subspace E of dimension λn, there exists a convex body $T \subset E$ such that

$$d_{BM}(Proj_E(K), T) < c'(\lambda), L_T < C(\lambda, A) \dots\dots (8)$$

Where $Proj_E$ is the orthogonal projection onto E in $\mathbb{R}^n$, and $c'(\lambda), C(\lambda, A)$ are independent of K and of n.

Embedding also serves to construct a section of anybody in the space $\mathbb{R}^n$.

Theorem: Let the space $X = (\mathbb{R}^2, \|.\|, |.|)$ have the following property: $\|x\| \leq |x|$ for all $x \in \mathbb{R}^n$ and for some $C > 0$ and $t > 0$, for every integer k, $\frac{n}{2} \leq k \leq n$, there are subspaces $S_k \in G_{n,k}$ such that

$$|x| \leq C\left(1 - \left(\frac{k}{n}\right)\right)^{-t} \|x\| \dots\dots (9)$$

For all $x \in S_k$, and moreover, for every such that k, the probability that our element of $G_{n,k}$ satisfies this inequality is at least $1 - \left(\frac{1}{n}\right)$. Then there is a $V = V(C, t)$ such that $\left(|B_{\|.\|}| / |B_{|.|}|\right)^{1/n} \leq V$.□

Corollary: *Let $K \subseteq \mathbb{R}^n$ be a body with $L_K < A$, and let $0 < \varepsilon < 1$. Then for any subspace E of dimension εn, there exists a convex body $T \subseteq E$ such that*

$$c(1-\varepsilon)proj_E(K) \subseteq T \subseteq C(1+\varepsilon)proj_E(K) \ldots\ldots (10)$$

Then there exist $k = k(n,\varepsilon)$ points such that:

$$\|T\| \leq C(\varepsilon)\sqrt{k}|proj_E(K)| \ldots\ldots (11)$$

Then, with probability greater than $1 - 2e^{-\frac{k(C(\varepsilon))^2}{L_T^2}}$

$$R(T) \leq C(\varepsilon)\sqrt{k}R(proj_E(K)) \ldots\ldots (12)$$

Concentration of measure Verses Large Deviation Principle

The spirit of large deviation takes a chance in the theory of concentration of measure. The large deviation principle demands rate function which controls the speed of convergence (neighborhood). In [9] we found that $A_\varepsilon = \{x \in S^n; d(x,A) < \varepsilon\}$ is the neighborhood of order $\varepsilon > 0$ of A for the geodesic metric on S^{n-1}. The function A_ε can be control the concentrate phenomenon. On the same area if f is continuous on S^n with modulus of continuity $\omega_f(\delta) = \sup\{|f(x) - f(y)|; d(x,y) < \delta\}$then

$$\sigma^n\left(\left|f - m_f\right| \leq \omega_f(\delta)\right) \leq 2e^{-(n-1)\delta^2} \ldots\ldots (13)$$

Where m_f is the median of the function f.

The concentrate phenomenon serves also to generate sections of anybody in the space $\mathbb{R}^n$. Concentration of measure determined a probability measure in such way that (X, d, μ) is a metric space which equipped with the Borel measure μ. The level of concentration is determined with respect to the class of linear functional ($lipschitz - function$) by measuring the size of minimal well – distributed substructures of certain probability space. And these substructures should exhibits a high level of concentration and, at the same time, they should represent the original space in an essential way.

Definition: A sequence of random variable $X_1, X_2, \ldots$, with values in metric space is said to satisfy a large deviation principle with

Speed $a_n \to \infty$ and

Rate function I,

If, for all Borel sets $A \subset M$

$$\lim_{n\to\infty} \frac{1}{a_n} \log P\{X_n \in A\} \leq -\inf_{X \in cl(A)} I(X)$$

$$\lim_{n\to\infty} \frac{1}{a_n} \log P\{X_n \in A\} \geq -\inf_{X \in int(A)} I(X)$$

Lemma: If I is a rate function and A is a Borel set, such that for every $x \in A$ and $\varepsilon > 0$ with $B(x,\varepsilon) \subset A$.

$$\lim_{n\to\infty} \frac{1}{a_n} \log P\{X_n \in B(x, \varepsilon)\} \geq -I(X) \ldots\ldots (14)$$

Then,

$$\lim_{n\to\infty} \frac{1}{a_n} \log P\{X_n \in A\} \geq -\inf_{x\in int\, A} I(X) \ldots\ldots (15)$$

So, for $B(x, \varepsilon) \subset A$, then $d_{BH}(B(x, \varepsilon), A) \leq C(\varepsilon)$ with $L(A) < C$.

The right hand side of (15) above stands for some known quantity in isoperimetric process. By the other hand the nature of $lipschitz\ functions$ appear in the sense of large deviation principle as contraction principle.

Lemma (Contraction Principle): If $X_1, X_2, \ldots$ satisfies a large deviation principle with speed a_n and good rate function I, and $f: M \to M'$ is continuous mapping, then the sequence $f(X_1), f(X_2), \ldots$ satisfies a large deviation principle with speed a_n and good rate function J given by

$$J(y) = \inf_{X\in f^{-1}(y)} I(X) \ldots\ldots (16)$$

Corollary: *If I is a rate function and $A$is a Borel set, such that for every $x \in A$ with $B(x, \varepsilon) \subset A$, then*

$$\left.\begin{array}{l} P\{X_n \in B(x,\varepsilon)\} \geq Ce^{-a_n I(x)}\ as\ n\to\infty,\ and \\ P\{X_n \in A\} \geq Ce^{-a_n \inf_{x\in int\, A} I(x)}\ as\ n\to\infty \end{array}\right\} \ldots\ldots (17)$$

The rate function $I(x)$ here stands for $\frac{(\Delta x)^2}{(\sigma)^2}$. From Equation (17) we obviously see the notion of concentration of measure in the sense that $cB \subseteq A \subseteq CB$. Concentration of measure is used to prove that ε holds with high probability.

Definition: Let (X, d, μ) be a metric space with metric d and $diam(X) \geq 1$, which is equipped with Borel probability measure μ. Then the concentration function on X is (Isoperimetric constant) $\alpha(X; \varepsilon) = 1 - \inf\left\{\mu(A_\varepsilon): A\ Borel\ subset\ of\ X, \mu(A) \geq \frac{1}{2}\right\}$, where $A_\varepsilon = \{x \in X: d(x, A) \leq \varepsilon\}$ is the $\varepsilon - extension$ of A.

We had that $x \in A$ and this imply that $d(x, A) \leq \varepsilon$. So $\alpha(X; \varepsilon) \approx P(x \in X) = Ce^{-a_n \inf_{x\in int(X)} I(x)}$.

Now according to Theorems (1.6), (1.12) and Corollary (2.4) we had the following corollary.

Corollary: *Let $C > 0$, $\varepsilon > 0$. And let $X = (\mathbb{R}^n, \|.\|, |.|)$ be $n - dimensional\ normed\ space$ which have the following property: $\|x\| \leq |x|$ for all $x \in \mathbb{R}^n$. Let $B(x, \varepsilon) \subseteq X$ be a body in X. Then for $k \geq c(\varepsilon) \log n$ there exist a $k - dimensional$ subspace $S_k \in G_{n,k} \subseteq B(x, \varepsilon)$ such that:*

$$\|x\| \leq C\sqrt{k}(1 + \varepsilon)|x| \ldots\ldots (18)$$

For all $x \in B(x, \varepsilon)$, with probability greater than $1 - Ce^{-a_n inf_{x \in int\, A} I(x)}$, where a_n stands for speed of convergence and $I(x)$ stands for rate function.

Theorem (Concentration on S^{n-1}): Let $f: S^{n-1} \to \mathbb{R}$ be a $L_{S^{n-1}} - Lipschetz$ function. Then for any $\varepsilon > 0$

$$P\{|f(x) - Ef(x)| > L_{S^{n-1}}\varepsilon\} \leq 2\exp\left(-\frac{C\varepsilon^2 n}{L_{S^{n-1}}}\right) \dots\dots (19)$$

So we can deal with convex body $K \subseteq \mathbb{R}^n$.

Convex body & Diameter of its Section

By a convex body we mean isotropic convex body which is symmetric and centered at the origin. We say that a body $K \subset \mathbb{R}^n$ is convex if for $x, y \in K, \delta \in (0,1)$, the relation $\{\delta x + (1 - \delta)y\} \in K$ holds. Apostolos, Librini and Antonios in their paper [13] recall that $r(K)$ and $R(K)$ as inradius and radius alternatively for the convex body K with $0 \in int\,(K)$. Kannan, Lovasz and Simonovitz [14] had proved that $R(K) \leq (n + 1)L_K$, where L_K is the isotropic constant for the convex body.

Definition: A convex body in $\mathbb{R}^n$ is a compact convex subset K of $\mathbb{R}^n$ with nonempty interior.

The best method to describe the convex body is with its support function which is principle tool to characterize the convex body.

Definition: Let K be a convex body of volume 1 in $\mathbb{R}^n$, for $q \geq 1$ we define the $L_q - centroid$ body $Z_q(K)$ of K by its support function

$$h_{L_q(\mu)} \coloneqq \|\langle .\,, y\rangle\|_{L_q(\mu)} = I_{Z_q(K)}(y) = I_q(K, y) = \left(\int_K |\langle x, y\rangle|^q \,\mu(dx)\right)^{\frac{1}{q}} \dots\dots (20)$$

Fact: we know that the $q - th$ moment of any body for $q \geq 1$ will be at the form:

$$I_q(K) = \left(\int_K \|x\|_2^q \,dx\right)^{1/q} \dots\dots (21)$$

$$I_q(x, y) = \left(\int_K |\langle x, y\rangle|^q\right)^{\frac{1}{q}} \dots\dots (22)$$

Theorem (Alesker): Let K be an isotropic convex body in $\mathbb{R}^n$. For $q \geq 2$ we have

$$I_q(K) \leq C\sqrt{q} I_2$$

Where $C > 0$ is an absolute constant.

Also this moment function guarantees the embedding assumption

Lemma: There exist an absolute constants $C_1, C_2 > 0$ such that

$$C_1 L_\mu \leq \left(\int |\langle x,y\rangle|^p \mu(dx)\right)^{\frac{1}{p}} \leq C_2 C_\alpha \max\left\{1, p^{\frac{1}{\alpha}}\right\} L_\mu \dots\dots (23)$$

For every $p > 0$and $y \in S^{n-1}$.

We can conclude and according to Theorem (2.7) that

Corollary: *Let $K \subseteq \mathbb{R}^n$ be an isotropic convex body with log – concave measure μ, set $\varepsilon \in (0,1)$ and let $S^{n-1} \subseteq K$, with*

$$cL_{S^{n-1}} \leq \left(\int_K |\langle x,y\rangle|^p \mu d(x)\right)^{\frac{1}{p}} \leq CC_\varepsilon\, max\left[1, p^{\frac{1}{\varepsilon}}\right] L_{S^{n-1}} \dots\dots (24)$$

Then with probability greater than $Ce^{-n\frac{C_3^2 \max\left[1,p^{\frac{1}{3}}\right]}{L_{S^{n-1}}}}$, we have

$$\|\langle x,y\rangle\|_{L_p(\mu)} \leq CL_{S^{n-1}}(p)^{\frac{1}{\varepsilon}} \dots\dots (25)$$

For $p > 0$ and $y \in S^{n-1}$.Where c, C are universal constants.

So, $L_{S^{n-1}}$ will control the neighborhood process.

The most advantage of the isotropic convex body with centered at the origin is that it does create a convex hull according to its measure.

Fact: We say that the convex body $K \subseteq \mathbb{R}^n$ is isotropic if $|K| = 1$and $\left(\int_K (|\langle x,y\rangle|^2)\ \ \mu(dx)\right)^{\frac{1}{2}} = L_K$. Where L_k is the isotropic constant

Theorem: Let $\varepsilon \in (0,1)$ and K be an isotropic convex body with centroid at the origin in $\mathbb{R}^n$. For every $\delta \in (0,1)$, $m = \exp(\varepsilon n)$ points $x_1, \dots, x_m$ chosen uniformly and independently form K satisfy with probability greater than $1 - \delta$

$$K \supset co(x_1, \dots, x_m) \supset C(\delta)\varepsilon K \dots\dots (26)$$

Also, the notion of convex body aids in embedding process.

Theorem: For a centrally – symmetric convex body $K \subset \mathbb{R}^n$, there exists a centrally – symmetric convex body $T \subset \mathbb{R}^n$ with $d_{BM}(K,T) \leq C_1 \log n$ and $L_T < C_2$, where $C_1, C_2 > 0$ are numerical constant.

So, with probability greater than $Ce^{-n\frac{C_1^2\varepsilon^2}{L_T}}$, we had $\|x\| \leq C_1 \varepsilon L_T$. Where L_T is the convex constant

As in Theorem (3.6) the random points needs to distribute uniformly and independently and that will necessity a comfortable Borel distribution measure, which relevant in some way to notion of concentration of measure

Theorem: Let S^n be a unite sphere in $\mathbb{R}^{n+1}$ equipped with a geodesic d, let z distributed according to μ^n on S^n. Let A be a measurable set of S^n that satisfies $\mu^n(A) \geq \frac{1}{2}$,and $A_\varepsilon = \{x \in S^n: d(x,y) \leq \varepsilon \, for \, some \, y \in A\}$. Then $P(z \in A_\varepsilon) = \mu^n(A_\varepsilon) \geq 1 - e^{-\frac{(n-1)\varepsilon^2}{2}}$.

So, to get the notion of neighborhood, we need a probability distribution function, and for the convex body this function is considered as the support function as in Definition (3.2)

The other advantage of the convex body is that it creates a measure of its own.

Theorem: Every convex body K creats a log – concave measure μ_K, and random set of $\exp(\varepsilon n)$ points chosen from K creates a body equivalent to K and, at the same time, form a $\Psi - distribution$ for μ_K.

Definition: We say the Borel probability measure μ is log – concave if $\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}$, and

μ is isotropic if $\int_{\mathbb{R}^n} \langle x, y \rangle^2 \mu(dx) = L_\mu^2 \; \forall y \in S^{n-1}$.

μ is satisfy a $\Psi_\alpha - estimate$with constant $C_\alpha \geq 1$ if $\|\langle x, y \rangle\|_{L_{\Psi_\alpha}(\mu)} \leq C_\alpha L_\mu \forall y \in S^{n-1}$.

Fact: Every convex body can be identified with its polar body $K^0 \coloneqq \{y \in \mathbb{R}^n: \forall x \in K, \langle x, y \rangle \leq 1\}$.

So, according to Ex – fact, we can concentrate the convex body around its polar body, which is isotropic convex body, to identified it, i.e. $K^0 \subseteq K$. If we back to Lemma (1.1) there exists $x_1, \dots, x_k \in K$which are orthogonal with respect to $\langle . , . \rangle_{K^0}$ taking $\varepsilon = 1$, and that will built a net for K. As in Theorem (1.4) this k points will produce an embedding into ℓ_2^k which equivalent in some way to the polar body of K, with rate function as log – concave measure.

Corollary (Diameter of Convex Body According to its Polar Body):

Let $K \subseteq \mathbb{R}^n$ be an isotropic convex body with $K^0 \subseteq K$, let $\varepsilon = 1$ and $k = C \log n$, there exists a subspaces $E \subseteq K^0$ such that

$$cL_K \leq \left(\int \left(diam(K \cap E) \right)^p dx \right)^{\frac{1}{p}} \leq C \max[1, \sqrt{p}] \, L_K \dots \dots (27)$$

Then with probability greater than $1 - Ce^{-\frac{C^2 p}{L_K}}$ we have

$$\left(\int \left(diam(K \cap E) \right)^p dx \right)^{\frac{1}{p}} \leq C\sqrt{p} \left(\int \left(diam(K \cap K^0) \right)^2 dx \right)^{\frac{1}{2}} \dots \dots (28)$$

Then,

$$diam(K \cap E) \leq C\sqrt{p} L_K \dots \dots (29)$$

Discussion: We found that the best method to give approximated diameter for anybody in the space is with embedding it to its equivalent Euclidean body and compare its metric with the legal Euclidean metric.

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