Bounds and Inequalities for Thinning and Imperfect Repair Processes

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Abstract:

In this paper, we present inequalities for the average number of points retained in thinning process generated from ordinary renewal process with distribution function F and inequalities for imperfect repair process.The linear bounds are obtained for thinning process in the case of independent censures.

On the other hand, a numerical illustration is presented.

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1 Introduction

Some stochastic processes are widely studied in the literature of reliability such as thinning process and imperfect repair process. In the imperfect repair model, the repaired item is returned to the "as-good-as-new" state with probability $p(t)$ and to the imperfect one (a functioning state, but with age equal to its age at failure) with probability $1-p(t)$. Brown and Proschan (1983) and Block & al. (1981) provide extensive studies about such models, in particular preservation and monotonicity properties.

Many applications to system analysis, queueing theory and reliability problem motivate development of thinning model in which some of the points in the original process are deleted with probability $1-p$, and retained with probability p. The preservation properties of such process are studied by Kovats & Mori (1992). For details about this process, see Aissani (1997), Cox & Isham (1980) and Kalashnikov (1990).

In this paper, we give in theorem 4 linear bounds for the average number of points retained in thinning process generated by ordinary renewal process with distribution function F for the case of independent censures based on the classic renewal theory (Barlow & Proschan (1975)). We give additional results of inequalities similar to the ones established by Ebrahim & Pellerey (1995) for the uncertainty measure for imperfect repair process.

Finally, we give a calculation example of the upper bounds values for the renewal function of the $HNBUE$ class in the thinning process for several values of p such as $p \in]0,1]$.

2 Notations and definitions

Let X be a non-negative random variable with cumulative distribution function $F(.)$, density function $f(.)$, survival function $\overline{F}(.) = 1 - F(.)$, failure rate $\lambda(.) = \dfrac{f(.)}{1-F(.)}$ and finite mean $\mu = E(X)$.

Definition 1 *F is IFR (Increasing Failure Rate) if*

$$\lambda(t) \text{ is non decreasing in } t, \qquad \forall t \geq 0.$$

F is $NBUE$ (New Better than Used in Expectation) if

$$\int_t^{\infty} \overline{F}(x)\,dx \leq \mu \overline{F}(t), \qquad \forall t \geq 0.$$

F is $HNBUE$ (Harmonic New Better than Used in Expectation) if

$$\int_t^{\infty} \overline{F}(x)\,dx \leq \mu \exp\left(\frac{-t}{\mu}\right), \qquad \forall t \geq 0.$$

The corresponding dual concepts namely DFR, NWUE, HNWUE are defined by changing the direction of the monotonicity or the inequality as appropriate. Here, D and W stand decreasing and worse, respectively.

Definition 2 *The uncertainty measure of a distribution F is the differential entropy,*

$$H(f) = -\int_0^{\infty} f(x) \log f(x)\,dx = -E(\log f(X)),$$

and is commonly referred to as the Shannon information measure. (see Shannon (1948))

Ebrahimi (1995) defined the uncertainty of residual lifetime distribution, $H(f;t)$, of a component by

$$H(f;t) = 1 - \frac{1}{\overline{F}(t)} \int_t^{\infty} f(x) \log \lambda(x)\, dx.$$

If the component has survived until time t, $H(f;t)$ measures the expected uncertainty in the conditional density of $X_t = X - t$ given that $X > t$ about the predictability of the remaining lifetime of the component.

Definition 3 *Let X and Y be two random variables with distribution functions $F(.)$ and $G(.)$, failure rates λ_F and λ_G and entropies $H(f;t)$ and $H(g;t)$ respectively.*

X is said to be smaller than Y in the distribution order (denoted $X \leq_{st} Y$) if

$$F(x) \geq G(x), \qquad \forall x \in \mathbb{R}.$$

X is said to be smaller than Y in the failure rate order (denoted by $X \leq_{FR} Y$) if

$$\lambda_F(x) \geq \lambda_G(x), \qquad \forall x \geq 0.$$

X is said to be smaller than Y in the increasing convex order (denoted by $X \leq_{icx} Y$ or $X \leq_v Y$) if

$$\int_x^{+\infty} \overline{F}(u)\, du \leq \int_x^{+\infty} \overline{G}(u)\, du, \qquad \forall x \geq 0.$$

X is said to have less uncertainty than Y (denoted $X \leq_{LU} Y$) if

$$H(f;t) \leq H(g;t), \qquad \forall t \geq 0.$$

Model 1: Thinning process

Let $X_1, X_2, ...$be a Bernoulli sequence of mutually independent random variables and $\{t_i\}$ an ordinary renewal process with probability distribution

$$F(x) = P\ \{t_i - t_{j-1} < x\}, \qquad -\infty < x < +\infty \text{ and } \forall j.$$

If Ψ denotes the number of successively censored points, then N_p is a renewal process with probability distribution

$$F_p(x) = \sum_{k=0}^{\infty} P(\Psi = k)\ F^{(k+1)}(x) = \sum_{k=0}^{\infty} p\ (1-p)^k\ F^{(k+1)}(x), \tag{3.1}$$

where $F^{(k)}(.)$ is the $k-$fold convolution of F with itself.

The mean of this process is

$$\mu_p = E(X_p) = \frac{\mu}{p}, \qquad \text{where } \mu = E(X).$$

Model 2: Imperfect repair process

Consider that an equipment is put in operation at time $t = 0$ and every time a failure occurs it is repaired.

If t is the equipment's age at failure, then with probability $p(t)$, it is restored to its good as new condition (complete repair) and with probability $q(t) = 1 - p(t)$, it is restored to its condition just prior to failure (minimal repair).

The intervals between successively perfect repair form an ordinary renewal process with inter-arrivals distribution

$$F_p(t) = 1 - \exp\left\{ -\int_0^t p(x)\ \overline{F}^{-1}(x)\ dF(x) \right\}, \qquad \forall t \geq 0.$$

If F has a failure rate $\lambda(t)$, then F_p has a failure rate $\lambda_p(t)$ such as

$$\lambda_p(t) = p(t)\,\lambda(t), \qquad t \geq 0,$$

and

$$\overline{F}_p(t) = \exp\left\{ -\int_0^t p(x)\ \lambda(x)\ dx \right\}.$$

3 Inequalities for thinning processes

Let $N_p(t)$ be a thinning point process of independent censures, with parameter p such that $0 < p < 1$, and a distribution function $F_p(x)$ given by (3.1).

We denote by $H_p(t) = E(N_p(t))$ the average number of successively censored points, and on the other hand we introduce the following notations

$$I_1(b,t) = \frac{t}{\mu_p} + b - b\overline{F}_p(t) - \hat{F}_p(t) + F_p(t), \tag{3.2}$$

$$I_n(b,t) = \frac{t}{\mu_p} + b - b\overline{F}_p^{(n)}(t) - \sum_{k=1}^{n} \hat{F}_p * F_p^{(k-1)}(t) + \sum_{k=1}^{n} F_p^{(k)}(t), \tag{3.3}$$

where $\hat{F}_p(t) = \frac{1}{\mu_p}\int\limits_0^t \overline{F}_p(u)du$ is the equilibrium (or excess) distribution of the residual time for thinning process, b is a constant and $\mu_p = \int\limits_0^\infty \overline{F}_p(u)du$ is the average interval between successively censored points.

Also, we define $A = \left\{t \geq 0 : \overline{F}_p(t) > 0\right\}$ and

$$b_l = \inf_{t\in A} \frac{F_p(t) - \hat{F}_p(t)}{\overline{F}_p(t)}\ ;\ b_u = \sup_{t\in A} \frac{F_p(t) - \hat{F}_p(t)}{\overline{F}_p(t)}. \tag{3.4}$$

Theorem 4 *(i) For all F_p,*

$$H_p(t) \geq \frac{t}{\mu_p} - \hat{F}_p(t),\ \textit{where}\ \mu_p = \frac{\mu}{p}.$$

(ii) If in addition F_p is $NBUE$, then

$$\frac{t}{\mu_p} - 1 \leq H_p(t) \leq \frac{t}{\mu_p}.$$

(iii) If F_p is $HNBUE$, then

$$H_p(t) \leq \left(\frac{t}{\mu_p}\right)\alpha(t), \qquad t \geq 0$$

where $\alpha(t)$ is the solution (for α) of the equation

$$\exp\left(\frac{\alpha t}{\mu_p}\right) - \left(\frac{\alpha t}{\mu_p}\right)\exp\left(\frac{t}{\mu_p}\right) = 1 + \left[\left(2 - \left(\frac{t}{\mu_p}\right)\right)\exp\left(\frac{t}{\mu_p}\right) - e\right] I^{(t)}_{(\mu_p,\infty)},$$

where $I^{(t)}_{(a,b)}$ denotes the indicator function of the interval (a,b).

(iv) If F_p is $HNWUE$, then

$$H_p(t) \geq \frac{t}{\mu_p\left[1 - \exp\left(\frac{t}{\mu_p}\right)\right]} - 1, \quad t \geq 0.$$

(v) For all

$$F_p, I_n(b_l,t) \leq H_p(t) \leq I_n(b_u,t), \qquad n = 0,1,2,...,$$

where $I_n(b_l,t)$ *is increasing in* n *for* $t \geq 0$*, and* $I_n(b_u,t)$ *is decreasing.*

Also,

$$H_p(t) = \lim_{n\to\infty} I_n(b,t) \ \ \textit{for any real}\, b,$$

and for $b_l < b < b_u$*, we have*

$$\begin{array}{ll} I_0(b,t) < I_1(b,t), & \textit{for some}\, t, \\ I_0(b,t) > I_1(b,t), & \textit{for some}\, t. \end{array}$$

In particular, the first possible encadrement is

$$\frac{t}{\mu_p} + b_l \leq H_p(t) \leq \frac{t}{\mu_p} + b_u.$$

Proof. $N_p(t)$ is an ordinary renewal process because it is generated by an ordinary renewal process, having a distribution function $F_p(t)$ and so $H_p(t)$ satisfies the integral renewal equation:

$$H_p(t) = F_p(t) + \int_0^t H_p(t-u)dF_p(u)\ , \quad t \geq 0. \tag{3.5}$$

The lower bound in *(i)* is better than the bound $\dfrac{t}{\mu_p} - 1$ (because $\hat{F}_p(t) \leq 1$) given in the theorem 3.14 Page 171 for Barlow & Proschan [3].

First, we prove that $\dfrac{t}{\mu_p} - 1$ is well a lower bound of $H_p(t)$.

Let $N_p(t)$ be the number of renewals in the interval $(0,t)$ for the thinning process.

$$H_p(t) = E\left[N_p(t)\right] = \sum_{n=1}^{\infty} F_p^{(n)}(t).$$

We have

$$t \leq t_{N_p(t)+1} = X_1^{(p)} + X_2^{(p)} + \ldots + X_{N_p(t)+1}^{(p)}.$$

Thus

$$t \leq \sum_{i=1}^{N_p(t)+1} X_i^{(p)}.$$

The random variables $X_1^{(p)}, X_2^{(p)}, \ldots, X_{N_p(t)}^{(p)}$ are independent from the event $\{N_p(t)+1\}$, then using *Wald* identity, we find:

$$t \leq E\ [N_p(t)+1\ ]\, E\left(X_i^{(p)}\right).$$

Thus

$$H_p(t) \geq \frac{t}{\mu_p} - 1, \quad \forall t \geq 0. \tag{3.6}$$

(i) To prove *(i)*, it is sufficient to substitute the last inequality (3.6) in the renewal equation, and then we find:

$$\begin{aligned} H_p(t) &\geq F_p(t) + \int_0^t \left[\frac{(t-u)}{\mu_p} - 1\right] dF_p(u) \\ &= \frac{t}{\mu_p} F_p(t) - \frac{1}{\mu_p}\int_0^t u\ dF_p(u) \\ &= \frac{t}{\mu_p} - \frac{1}{\mu_p}\int_0^t \overline{F}_p(u)du \\ &= \frac{t}{\mu_p} - \hat{F}_p(t). \end{aligned}$$

(ii) We assume that F_p is $NBUE$. Then, by definition we have:

$$\hat{F}_p(t) = \frac{1}{\mu_p}\int_0^t F_p(u)du \geq F_p(t)\ , \quad \forall t \geq 0.$$

But

$$\begin{aligned} \hat{H}_p(t) &= \sum_{n=1}^{\infty}\int_0^t \hat{F}_p(t-u)\ dF_p^{(n-1)}(u) \\ &\geq \sum_{n=1}^{\infty}\int_0^{\infty} F_p\,(t-u)\ \ dF_p^{(n-1)}(u) = H_p(t). \end{aligned}$$

From theorem 3.8 of Barlow & Proschan [3], we have:

$$\hat{H}_p(t) = \frac{t}{\mu_p} = \frac{pt}{\mu}, \tag{3.7}$$

and hence,

$$H_p(t) \leq \frac{t}{\mu_p}. \tag{3.8}$$

From (3.7) and (3.8), we can write the following result:

$$\text{If } F_p \text{ is } NBUE \text{ then } \frac{t}{\mu_p} - 1 \leq H_p(t) \leq \frac{t}{\mu_p}\ , \quad \forall t \geq 0.$$

Using the bound $H_p(t) \geq \dfrac{t}{\mu_p} - \hat{F}_p(t)$ and substituting it in the renewal equation, we find after n iterations:

$$H_p(t) \geq \frac{t}{\mu_p} + \sum_{k=1}^{n} F_p^{(k)}(t) - \sum_{k=1}^{n} \hat{F}_p * F_p^{(k-1)}(t) - F_p^{(n)}(t) \quad , n \geq 0, \tag{3.9}$$

where $F_p^{(0)}(t) = 1$, and the summations are taken to be zero if $n = 0$.

The sequence of lower bounds defined by (3.9) is monotone non decreasing in n for any fixed t, and converges to $H_p(t)$.

Indeed, from [3]

$$H_p(t) = \sum_{k=1}^{\infty} F_p^{(k)}(t). \tag{3.10}$$

$$\frac{t}{\mu_p} = \sum_{k=1}^{n} \hat{F}_p * F_p^{(k-1)}(t). \tag{3.11}$$

$$\lim_{n \to \infty} F_p^{(n)}(t) = 0. \tag{3.12}$$

Monotonicity is proved by induction. We have seen previously that

$$H_p(t) \geq \frac{t}{\mu_p} - \hat{F}_p(t) \geq \frac{t}{\mu_p} - 1.$$

Consequently, the property holds for $n = 1$. Assume for some $n > 1$ that

$$\sum_{k=1}^{n} F_p^{(k)}(t) - \sum_{k=1}^{n} \hat{F}_p * F_p^{(k-1)}(t) - F_p^{(n)}(t) \geq \sum_{k=1}^{n-1} F_p^{(k)}(t) - \sum_{k=1}^{n-1} \hat{F}_p * F_p^{(k-1)}(t) - F_p^{(n-1)}(t).$$

Convolving both sides of this inequality with $F_p\,(t)$ and adding $F_p(t) - \hat{F}_p(t)$ to each side gives the desired result.

(iii) Use X_p^* to represent the random variable exponentially distributed with the same mean as X_p i.e. $F_p^*(x) = 1 - \exp\left\{\dfrac{-x}{\mu_p}\right\}$, $x > 0$.The variable

$\hat{X}_p$ denotes the random variable with distribution $\hat{F}_p(x) = \dfrac{1}{\mu_p} \int\limits_0^x \overline{F}_p(t)\, dt$.

By definition, X_p is $HNBUE$ $(HNWUE)$ if and only if $X_p^* \geq_{icx} (\leq_{icx}) X_p$.

Let α be any number greater than 1. Hence H_p is non-decreasing and F_p is $HNBUE$, it follows from (3.7) that, for $t > 0$,

$$H_p(t) = \frac{\alpha t}{\mu_p} \quad = \hat{F}_p(\alpha t) + \int\limits_0^{\alpha t} H_p(\alpha t - x) \; d\hat{F}_p(x)$$

$$= \left[1 - \exp\left(\frac{-\alpha t}{\mu_p}\right)\right] + \int\limits_0^{\alpha t} H_p(\alpha t - x)\, \frac{1}{\mu_p} \exp\left(\frac{-x}{\mu_p}\right) dx,$$

or

$$\alpha t \geq \mu_p \left[1 - \exp\left(\frac{-\alpha t}{\mu_p}\right)\right] + \exp\left(\frac{-\alpha t}{\mu_p}\right) \int\limits_0^{\alpha t} H_p(y) \exp\left(\frac{y}{\mu_p}\right) dy. \tag{3.13}$$

It follows that for any $t > 0$:

$$\alpha t \geq \mu_p \left[1 - \exp\left(\frac{-\alpha t}{\mu_p}\right)\right] + \mu_p \; H_p(t) \left[1 - \exp\left(\frac{-(\alpha - 1)\, t}{\mu_p}\right)\right].$$

Hence

$$H_p(t) \leq \inf_{\alpha > 1} g_p(\alpha) \quad , t > 0,$$

where $g_p(\alpha) = \dfrac{\alpha t - \mu_p \left[1 - \exp(\frac{-\alpha t}{\mu_p})\right]}{\mu_p \left[1 - \exp(\frac{-(\alpha-1)t}{\mu_p})\right]}$.

Elementary calculus shows that the infimum is attained at $\alpha = \alpha_1$ where $\alpha_1 = \alpha_1(t)$ is the solution to

$$\exp\left(\frac{\alpha t}{\mu_p}\right) - \left(\frac{\alpha t}{\mu_p}\right) \exp\left(\frac{t}{\mu_p}\right) = 1$$

The infimum equals $\left(\dfrac{t}{\mu_p}\right) \alpha_1(t)$ which is, therefore, an upper bounds of $H_p(t)$ for all $t > 0$.

However, if $t > \mu_p$, a sharper upper bound is obtained as follows:

If $t > \mu_p$ then

$$\begin{aligned}\int_0^{\alpha t} H_p(y)\exp\left(\frac{y}{\mu_p}\right)dy &\geq \int_{\mu_p}^{t} H_p(y)\exp\left(\frac{y}{\mu_p}\right)dy + \int_t^{\alpha t} H_p(y)\exp\left(\frac{y}{\mu_p}\right)dy \\ &\geq \int_{\mu_p}^{t}\left(\frac{y}{\mu_p}-1\right)\exp\left(\frac{y}{\mu_p}\right)dy + H_p(t)\int_t^{\alpha t}\exp\left(\frac{y}{\mu_p}\right)dy \\ &= \left(t-2\mu_p\right)\exp\left(\frac{t}{\mu_p}\right) + \mu_p H_p(t)\left[\exp\left(\frac{\alpha t}{\mu_p}\right) - \exp\left(\frac{t}{\mu_p}\right)\right].\end{aligned}$$

It follows from (3.13) and the above that

$$H_p(t) \leq \inf_{\alpha>1} h_p(\alpha),$$

where

$$h_p(\alpha) = \frac{\alpha t - \mu_p - \mu_p(e-1)\exp\left(\frac{\alpha t}{\mu_p}\right) - \left(t-2\mu_p\right)\exp\left(\frac{-(\alpha-1)t}{\mu_p}\right)}{\mu_p\left[1-\exp\left(\frac{-(\alpha-1)t}{\mu_p}\right)\right]}.$$

Computations show that, for $t > \mu_p$, the function $h_p(\alpha)$, $\alpha > 1$, attains its infimum $\left(\frac{t}{\mu_p}\right)\alpha_2(t)$ at $\alpha_2(t)$, where α_2 is the solution of the equation:

$$\exp\left(\frac{\alpha t}{\mu_p}\right) - \left(\frac{\alpha t}{\mu_p}\right)\exp\left(\frac{t}{\mu_p}\right) = 1 - e + \left(2 - \frac{t}{\mu_p}\right)\exp\left(\frac{t}{\mu_p}\right).$$

The theorem follows on defining $\alpha(t)$ as $\alpha_1(t)$ or $\alpha_2(t)$ according to whether $0 \leq t \leq \mu_p$ or $t > \mu_p$.

(iv) The lower bound according to the case where F_p is $HNWUE$ can be easily improved. It is clear from (3.7) that

$$\frac{t}{\mu_p} \leq \hat{F}_p(t) + H_p(t)\hat{F}_p(t) = \hat{F}_p(t)\ \left[1 - H_p(t)\right],$$

so that, when F is $HNWUE$,

$$H_p\left(t\right) \geq \frac{t}{\mu_p\left[1-\exp\left(-\frac{t}{\mu_p}\right)\right]} - 1, \quad t \geq 0.$$

(v) We prove that $I_n(b_l, t)$ and $I_n(b_u, t)$ provided the best linear lower and upper bounds which are in a certain sense "best" when n is great and converges to $H_p(t)$ as $n \to \infty$.

Note that

$$b_l \leq \frac{F_p(t) - \hat{F}_p(t)}{\overline{F}_P(t)} \leq b_u, \qquad\qquad t \in A.$$

Therefore

$$b_l \overline{F}_p(t) \leq F_p(t) - \hat{F}_p(t) \leq b_u \overline{F}_p(t).$$

Convolve with $F_p^{(n)}(t)$ to get

$$b_l \overline{F}_p * F_p^{(n)}(t) \leq F_p^{(n+1)}(t) - \hat{F}_p * F_p^{(n)}(t) \leq b_u \overline{F}_P * F_p^{(n)}(t).$$

Now we sum over n to get

$$b_l \leq H_p(t) - \frac{t}{\mu_p} \leq b_u,$$

which proves *(v)* for $n = 0$.

Successive iterations in the right-hand side of (3.5) prove *(v)* for all n.

We prove *(v)* in a one sense inequality (lower bound), the other inequality follows in the same way.

Assume that

$$H_p(t) \geq \frac{t}{\mu_p} + b_l - b_l \overline{F}_p^{(n)}(t) - \sum_{k=1}^{n} \hat{F}_p * F_p^{(k-1)}(t) + \sum_{k=1}^{n} F_p^{(k)}(t), \tag{3.14}$$

which is true at n order and we prove it at $(n+1)$ order. We substitute (3.14) in (3.5).

To prove that $I_n(b_l, t)$ is non-decreasing in n, we see that

$$b_l \leq b_l F_p(t) + F_p(t) - \hat{F}_p(t),$$

thus

$$I_0(b_l, t) = \frac{t}{\mu_p} + b_l \leq \frac{t}{\mu_p} + b_l - b_l \overline{F}_p(t) + F_p(t) - \hat{F}_p(t) = I_1(b_l, t).$$

Assume for some $n > 1$ that $I_{n-1}(b_l, t) \leq I_n(b_l, t)$. Convolving both sides of this inequality with $F_p(t) - \hat{F}_p(t)$ to each side gives $I_n(b_l, t) \leq I_{n+1}(b_l, t)$.

Hence, we obtain that $I_n(b_l, t)$ is non-decreasing in n. In the same way we prove that $I_n(b_u, t)$ is non increasing in n.

Now, if $b < b_u$, then for some t, $b < \dfrac{F_p(t) - \hat{F}_p(t)}{\overline{F}_p(t)}$ with $\overline{F}_p(t) > 0$.

Therefore, $b < bF_p(t) + F_p(t) - \hat{F}_p(t)$, and so $I_0(b,t) < I_1(b,t)$.The other inequality follows in the same way.

The convergence of the sequence of bounds I_n for all b follows from the conditions (3.10), (3.11) and (3.12).

In particular, we have $I_n(b_l, t) \leq H_p(t) \leq I_n(b_u, t)$ or

$$\frac{t}{\mu_p} + b_l \leq H_p(t) \leq \frac{t}{\mu_p} + b_u.$$

■

Remark 5 *The constants* b_l *and* b_u *can have other interpretations which are the lower and upper bounds respectively of the mean residual time of the distribution* F_p *of the interval between successively censored points of the thinnig process, so that*

$$b_l = \inf_{t \in A} \left[\frac{\overline{\hat{F}}_p(t)}{\overline{F}_p(t)} \right] - 1 = \inf_{t \in A} \frac{1}{\mu_p} \int_t^{\infty} \frac{\overline{F}_p(u)}{\overline{F}_p(t)} du - 1,$$

and

$$b_u = \sup_{t \in A} \frac{1}{\mu_p} \int_t^{\infty} \frac{\overline{F}_p(u)}{\overline{F}_p(t)} du - 1.$$

4 Inequalities for imperfect repair process

Theorem 6 *Let* $\{N_p(t)\}$ *be an imperfect repair process with parameter* $p(t)$ *and an underlying distribution* F*, then*

(i) If F *is* DFR *and* $p(t)$ *is an increasing function, then* $X_p \leq_{LU} X$*, i.e.* $H(f_p; t) \leq H(f; t)$.

(ii) If $p(t)$ *is increasing in* t *then* $H(f_p; t) - H(f; t)$ *is increasing in* t*, for all* $t \geq 0$.

(iii) If $N_{p'}(t)$ *is an imperfect repair process with parameter* $p'(t)$ *and an underlying distribution* F' *such as:*

(a) $p \leq p'$, *i.e.* $p(t) \leq p'(t)$, $\forall t \geq 0$;

(b) $p(x)/p'(x)$ *is increasing in* x;

(c) F *is* DFR;

(d) $F \leq_{FR} F'$.

Then $X_p \leq_{LU} X_{p'}$.

Proof. To prove theorem 6, we need the following result.

Lemma 7 *Let* X *and* Y *be the lifetimes of two components with survival functions* $\overline{F}$ *and* $\overline{G}$*, failure rates* λ_F, λ_G *and entropies* $H(f;t)$ *and* $H(g;t)$ *respectively. If*

(i) $\dfrac{\lambda_F(x)}{\lambda_G(x)}$ *is a non-decreasing function in* x;

(ii) F *is* DFR;

(iii) $X \leq_{FR} Y$.

Then it follows that $X \leq_{LU} Y$.

• To prove the theorem, we put $F = F_p$ and $G = F$ in the lemma. The hypothesis *(ii)* of the lemma is verified by assumption.

Also, $\dfrac{\lambda_{F_p}(x)}{\lambda_F(x)} = \dfrac{p(x)\lambda(x)}{\lambda(x)} = p(x)$ is non-decreasing function in x by assumption. Thus, the condition *(i)* of the lemma follows.

In the same way, the condition *(iii)* of the lemma is verified since $0 \leq p(x) \leq 1, \forall x$ and $\lambda_p(x) = p(x)\lambda(x)$; then $X_p \leq_{FR} X$.

From lemma 7, we get that $X_p \leq_{LU} X$, which proves the part *(i)* of the theorem.

• From point *(i)* of theorem 6, we have $X_p \leq_{LU} X$, i.e. $H(f_p;t) \leq H(f;t)$.

Moreover, $X_p \leq_{FR} X$ since $\lambda_p(x) = p(x)\lambda(x) \leq \lambda(x), \forall x$ [because $p(x) \leq 1$].

Therefore,

$$\begin{aligned} H'(f_p;t) &= \lambda_p(t)\left[Log\lambda_p(t) + H(f_p;t) - 1\right] \\ &\leq \lambda(t)\left[Log\lambda(t) + H(f;t) - 1\right] = H'(f;t), \end{aligned}$$

which proves the part *(ii)* of theorem 6.

• In the rest, we have just to prove the part *(iii)* of theorem 6.

We remark that the condition *(ii)* of the lemma 7 is verified by assumption.

However $\lambda_{p'}(x) = p'(x)\lambda(x) \leq p(x)\lambda(x) = \lambda_p(x)$ follows from *(iii)-(a)* of theorem 6. Then $X_{p'} \leq_{FR} X_p$. We deduce the condition *(iii)* of the lemma 7.

We have $\dfrac{\lambda_{F_p}(x)}{\lambda_{F_{p'}}(x)} = \dfrac{p(x)\lambda(x)}{p'(x)\lambda(x)} = \dfrac{p(x)}{p'(x)}$ is increasing in x,this is hold from *(iii)-(b)* of theorem 6. Then the hypothesis*(i)* of the lemma 7 is verified.

From the lemma 7, we get $X_p \leq_{LU} X_{p'}$. This completes the proof. ■

5 Numerical illustration

We give the upper bound value of the renewal function of a non parametric distribution $HNBUE$ in the case of a thinning process for different values of p such as $p \in]0, 1]$, on the basis of the result *(iii)* of theorem 3.

The upper bound values of $H_p(t)$ are given in table1 in the case $\dfrac{pt}{\mu} < 1$ and in table2 in the case $\dfrac{pt}{\mu} \geq 1$.

$\frac{t}{\mu}$ \ p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.019	0.039	0.059	0.079	0.099	0.118	0.138	0.158	0.177	0.196
0.2	0.039	0.079	0.118	0.157	0.196	0.235	0.237	0.312	0.349	0.387
0.3	0.059	0.118	0.177	0.235	0.292	0.349	0.406	0.462	0.517	0.572
0.4	0.079	0.158	0.235	0.311	0.387	0.462	0.536	0.609	0.681	0.753
0.5	0.099	0.196	0.292	0.387	0.480	0.572	0.663	0.753	0.841	0.928
0.6	0.118	0.235	0.349	0.462	0.572	0.681	0.788	0.894	0.997	1.001
0.7	0.138	0.273	0.406	0.536	0.663	0.788	0.911	1.032	1.1150	1.267
0.8	0.157	0.311	0.462	0.609	0.752	0.893	1.032	1.167	1.301	1.432
0.9	0.177	0.349	0.517	0.681	0.841	0.997	1.150	1.301	1.448	1.592
1.0	0.196	0.387	0.572	0.753	0.928	1.100	1.267	1.432	1.592	1.750

Table1

p / $\frac{t}{\mu}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
15	2.43	4.10	5.63	7.14	8.64	10.14	11.64	13.14	14.64	16.14
20	3.02	513	7.14	9.14	11.14	13.14	15.14	17.14	19.14	21.14
25	3.57	6.14	8.64	11.14	13.64	16.14	18.64	21.14	23.64	26.14
30	4.10	7.14	10.14	13.14	16.14	19.14	22.14	24.98	28.14	31.14
35	4.62	8.14	11.64	15.14	18.64	22.14	25.64	29.14	32.64	36.14
40	5.13	9.14	13.14	17.14	21.14	25.07	29.14	33.14	37.14	41.14
45	5.637	10.14	14.73	19.14	23.64	28.14	32.14	37.14	41.64	46.14
50	6.14	11.14	16.14	21.14	26.14	31.14	36.14	41.14	46.64	51.14
100	11.4	21.14	31.14	41.14	51.14	61.14	71.14	81.14	91.14	101.1
200	21.14	41.14	61.14	81.14	101.1	121.1	141.1	161.1	181.14	210.1

Table2

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