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A MODEL FOR GENERALIZED ASSIGNEMENT PROBLEM. BILINEAR CASE (BGAP) LOCAL AND GLOBAL APPROACH

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ABSTRACT: Despite recent increases in automation of production systems, the human factor has emerged as one of the most important aspects of a company's competitiveness in this changing environment. This evolution of the industrial world has involved a perpetual questioning of the production methods and in particular the methods of taking into account the operators. The rational use of human resources during the organization of activities becomes one of the concerns of companies.

Due to this fact, several researchers have been interested in human resources management, particularly in the problem of assignment. Behind this problem, there is actually a set of problems that vary according to the objective of optimization, constraints taken into account.

This paper exposes a solution for generalized assignment problem with bilinear form, it consist to minimize the cost of the assignments subjects to a set of specific constraints. The problem of bilinear programming was introduced by Konno [1]. Several practical cases can be modelled under this special form.

We proposed two successive approaches to solving the problem based on the equivalence between bilinear program and concave minimization using two distinct polyhedrons and the formal linearity of a dual problem formulation.

Keywords: Generalized Assignment Problem, Bilinear Program, Global and local search, Optimization.

1. INTRODUCTION:

The generalized assignment problem is a very popular NP-hard optimization problem; it was studied widely in the literature. This problem seeks the minimum cost assignments of n tasks to m agents such that each task is assigned to precisely one agent to capacity restrictions on the agents. The main problem start first by the position of a model, followed by some reformulations and a proposed algorithm, this problem is general, and take the following form:

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$$\min \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij} x_{ij}$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij} x_{ij} \le b_i \quad i = 1,\ldots\ldots, m$$

$$\sum_{i=1}^{m} X_{ij} = 1 \qquad j = 1,\ldots\ldots n \ \text{(task to be assigned)}$$

$$\sum_{j=1}^{n} \mathrm{a}_{ij} x_{ij} \le b_i \quad i = 1,\ldots\ldots m \ (\text{agent available})$$

$$x_{ij} \in \{0,1\} \ \text{for all } i, j$$

binary variable $x_{i,j} = \begin{Bmatrix} 1 & \text{if task } j \to i \\ 0 & \text{otherwise} \end{Bmatrix}$

where a_{ij} is the capacity used when task j is assigned to agent i

and b_i is the available capacity of agent i

C_{ij} : is the cost of assigning task j to agent i

Two approaches have been taken in this work, the first one consists to obtain a population of solutions by defining a region of attraction (all the set of points of solution), second propose an algorithm obtained by the solution of the first approach to produce an algorithm of global optimization. This solution supplied an effective tool and allows benefiting from advantages of both approaches.

The specificity of this paper is that several practical problems can be modelled by mathematical formulations called bilinear programming, (Vehicle routing, scheduling, facility rent, plane assignment, etc.). The reader can refer to the bibliography given at the end of the article.

2. RELATED WORK:

The formulation above was first studied by Srinivasan and Thompson [2] to solve a transportation problem. The term generalized assignment problem for this setting was introduced by Ross and Soland [3]. This model is a generalization of previously proposed model by DeMaio and Roveda [4] where the capacity absorption is agent independent $(i.e., a_{ij} = a_{j,} \ \forall i)$.

The classical assignment problem, which provides a one to one pairing of agents and tasks, can be solved in polynomial time [5]. However, in GAP, an agent may be assigned to multiple tasks ensuring each task is performed exactly once, and the problem is NP-hard [6]. Even the GAP with agent-independent requirements is an NP-hard problem [7].

The GAP has a wide spectrum of application areas ranging from scheduling[8] and computer networking[9] and facility location [10]. Nowakovski et al. [11] study the ROSAT space telescope scheduling where the problem is formulated as a GAP and heuristic methods are proposed. Multiperiod single-source problem (MPSSP) is reformulated as a GAP by Freling et al. [12]. Janak et al. [13] reformulate the NSF panel-assignment problem as a multiresource preference-constrained GAP. Other applications of GAP include lump sum capital rationing, loading in flexible manufacturing systems [14], *p*-median location [15], maximal covering location [16], and routing [17]. A summary of applications and assignment model components can be found in [18].

For a given instance, Konno and Gallot and Ullucci [19] presented an algorithm for the bilinear problem with linear constraints, the Konno approach use a sequence of flat cuts to generate a minimal local sequence of local minima for the problem , the other models of algorithms have been presented by Al Khayyal and Falk [20]. In 1985 Tuy [21] propose a global optimal method for a concave function around a convex polyhedron, Falk and Palocsay [22] presents an algorithm for the sum of fractional functions with linear constraints, in 1995. Quessada and Grossman [23] proposed a fractional bilinear problem.

3. MODEL WITH BILINEAR FORM:

We may begin our discussion of bilinear forms by looking at a special case that we are already familiar with. Given a vector space V over a field F, the dot product between two elements X and Y (represented as column vectors whose elements are in F) is the map $V \text{ X } F \to F$ defined by:

$$\langle X, Y \rangle = X^T.Y = x_1 y_1 + \dots\dots\dots + x_n y_n$$

The property of the dot product which we will use to generalize to bilinear forms is bilinearity; the dot product is a linear function from V to F if one of the elements is fixed.

The solution for generalized assignment problem with bilinear form consists to minimize the cost of the assignments subjects to a set of specific constraints. The principle of the solution can be proved by the implementation of algorithms which exploits the equivalence between bilinear program and the concave minimization and the generalized algorithm of Konno by using the polyhedral form mentioned below:

$$\begin{aligned}
&opt\ \ f(x,y) = px + x(Cy) + qy \\
&under\ \ x \in X\ \ ,\ \ y \in Y \\
&ou\ X, Y\ are\ two\ polyhedra\ not\ empty\ in \\
&R^n R^m \quad respectively \\
&X = \begin{Bmatrix} x \in R^n\ :\ Ax \le a, x \ge 0 \\ Y \in R^m\ :\ By \le b, y \ge 0 \end{Bmatrix} \\
&(P \in R^n, q \in R^m, C \in R^{nm})
\end{aligned}$$

This domain was developed by several researchers who specialized in various classes of problems. Numerous improvements to find the effective algorithms were proposed, The analysis of the structure of these classes allowed several techniques which gives a number of resolution methods, the best solutions are often supplied by methods of decomposition, heuristics, and the principle of enumeration Branch and Bound, the concept of cutting and equivalences between the linear problems and the bilinear problem .

In our future view, we are going to propose a new approach to resolve the generalized assignment problem, this approach consist to finding the minimum cost assignment of n job to m agents so that every job is allocated to exactly an agent.

4. FORMULATION

Let us consider a bilinear functions with respect to the variables x and y, the constraints are given by two polyhedron corresponding to x and y.

The form is:

$$opt\ \mathrm{f}(x,y)=\sum_{i=1}^{\mathrm{m}}C_i x_i+\sum_{j=1}^{n}d_j y_j+\sum_{i=1}^{m}\ \sum_{j=1}^{n}C_{ij}x_i y_j$$

$with\ \ x\in X$ and $y\in Y$ where

$$\sum_{i=1}^{m}a_i x_i\le a_0 \qquad x_{\mathrm{i}}\in\{0,1\}\quad i=1,.....,m \qquad (1)$$

$$\sum_{j=1}^{n}\mathrm{b}_{\mathrm{j}}y_j\le b_0 \qquad y_{\mathrm{i}}\in\{0,1\}\quad j=1,.....,n$$

The GAP with bilinear form was proposed where a situation either the practicable domain is defined by two separate sets and the objective function is bilinear, All x_i and y_i appear in the same constraint. It is known that (1) is a NP-hard problem with bilinear form and with respect to the variable x and y, the constraints are given by two convex polyhedrons corresponding respectively to x and y.

The method here consists to search where we can reduce the domain. When the domain becomes empty then a global solution is reached. In order to obtain a well-defined problem (1), we suppose that $f(x,y)$ is positive on $V(X)\ \mathrm{X}\ V(Y)$, the main result shows that to solve this problem, it is sufficient to consider the vertices of $V(X)$ and $V(Y)$. Therefore if the value of (1) is finite, then there exists a solution (x^*,y^*) such that x and y are vertices of $V(X)$ and $V(Y)$.

5. THE MODEL:

Obtain a set of solutions that we call "start solution" by searching for extreme points, this choice is justified to obtain a good choice of starting representing an intermediate solution which we call local minimum.

When obtaining this solution, we used the method called alternative choice by sweeping the entire practicable region to refine the solution and repair in a way the first solution to eliminate the conflicts, and finally propose an algorithm which will produce a global minimum.

Let us consider the bilinear objective function and two polyhedrons as set of constraints, the solution can be generated by considering only both polyhedron, we can easily verify in polynomial time that polyhedron X and Y are borned. If this case exists then the optimal value also exists such that:

$$x^*\in V(X)$$
$$y^*\in V(Y)$$

Indeed the infiniteness of $f(x)$ on X, if that was the case, it must be reached at $x^*\in V(X)$ where $f(x^*)=f(x,y)$ for all $y^*\in V(Y)$.

By interchanging the role of x and y, we obtain other forms of equivalence of the bilinear problem. Consequently and regarding the symmetric structure of the problem, it is clear that the necessary condition for the pair (x^*,y^*) with $x^*\in V(X)$ and $y^*\in V(Y)$ would be the optimal solution of the problem such as :

$$\min_{x\in X}\ f\ (x,y^*)=f(x^*,y^*)=\min_{y\in Y}\ f(x^*,y)\ .... \quad (2)$$

This condition is not sufficient, we have only:

(x^*,y^*) Satisfied (2) if $y^*=\arg\min\limits_{y\in Y}\ f\ (x^*,y)$ $(i,e.;y^*$ is the only minimizer $f(x^*,\bullet)$ on Y when x^* is an optimal solution of (3) which has the following form:

$$\min\{f(x):x\in X\},$$

$$\text{Where }\ f(x)=\inf\{f(x,y):y\in Y\}\quad ....\ ....\ .(3)$$

Or by symmetry

$$\min\{f(y) : y \in Y\},$$

$$\text{where } f(y) = \inf\{f(x,y) : x \in X\}$$

This argument is a property known for equivalence between the bilinear problem and the concave minimization problem. Consider now $f(x^*, y^*) \prec f(x^*, y)$ for all $y \in Y$ satisfying $y \neq y^*$ thus for all $y \in Y, y \neq y^*$, There is an open neighbourhood U_y of x^* satisfying $f(x^*, y^*) \prec f(x,y)$ for all $x \in U_y$.

Let $U = \cap\{U_y : y \in V(Y); y \neq y^*\}$ so that $x \in U$ we have: $f(x^*, y^*) \prec f(x,y) \forall y \in V(Y); y \neq y^*$

but

$$f(x^*, y^*) = \min_{x \in X} f(x, y^*),$$

Then, for all $x \in U$

$$f(x^*, y^*) \leq \min\{f(x,y) : y \in V(Y)\}$$
$$= \min_{y \in Y} f(x,y)$$

To obtain the pair (x^*, y^*) satisfying (2) we will exploit the basic principle illustrated by the method of Konno who summarizes the solution in the following stages:

1. let X, Y borned and let $x^1 \in V(X)$, put $k = 0$
2. Solve the linear program $\min\{(q + cx^k)y : y \in Y\}$ to obtain a vector y^k de Y such that $f(x^k, y^k) = \min_{y \in Y} f(x^k, y)$
3. Solve the linear program $\min\{(p + cTy^k)x : x \in X\}$ to obtain a vector y^{k+1} de X such that $f(x^{k+1}, y^k) = \min_{x \in X} f(x, y^k)$
4. if $f(x^{k+1}, y^k) = \min f(x^k, y^k)$ then stop, else put $k = k+1$ and go to step(2)

Seeing the finiteness of $V(X)^* V(Y)$, the situation $f(x^{k+1}, y^k) \prec \min f(x^k, y^k)$ do not can occur several times, consequently the procedure must stop after a number of iteration with the pair :

$f(x^k, y^k)$ such that $\min_{y \in Y} f(x^k, y) = f(x^k, y^k) = f(x^{k+1}, y^k) = \min_{x \in X} f(x, y^k)$.

5.1 PROPOSITION 1:

To obtain the problem (1) with positive $f(x, y)$ on X and Y, we need only to consider the both polyhedron X and Y. The solution (x^*, y^*) exists such that x^* and y^* are the two vectors of X and Y.

Let us note that $v \in \Re$, the value of the problem (1) and let $(x^*, y^*) \in X, Y$ for $f(x^*, y^*) = v$, the generated function $f(\bullet, y^*)$ where y is fixed becomes linear and consequently:

$\min\{f(x, y^*) : x \in X\}$ admit a vector x^* to be a solution, the inverse is also true. One of the properties of (1) is to observe also that $f(x, y)$ cannot be almost concave. An optimal solution (x^*, y^*) exists at an extreme point of (X, Y) where x^* and y^* are two extreme points of X and Y respectively.

This problem is very difficult to be solved, however it is relatively simple to be solved in the case where we searching an optimal solution of the extreme point, the most known solution is obtained by the alternative fixation of x and y to obtain the pair (x^*, y^*).

5.2 PROPOSITION 2:

By definition, we note the neighbourhood by a point x of $V(X)$ and y of $V(Y)$ **.** Let $V(X)$ and $V(Y)$ two vectors of X and Y respectively, an important property of the optimal solution is that if the bilinear problem has a finite value (if x and y are borned), then the optimal (x^*, y^*) exists such that $x \in V(X)$ and $y \in V(Y)$.

A local minimum is thus defined by a point (x^*, y^*) such that $f(x^*, y^*) \leq f(x, y)$ for all $x \in V(x^*)$ and $y \in V(y^*)$, where $V(x^*)$ and $V(y^*)$ are two sets of neighbouring extreme points to x^* and y^* .

An extreme point is neighbouring to (x^*, y^*) if and only if it takes the form (x^i, y^*) or $(x^\bullet, y^i)$

Where $x^i \in V(x^*), y^i \in V(y^\bullet)$.

Indeed the extreme point is a global minimum if $f(x^*, y^*) \leq f(x, y)$ for all $x \in \beta_\delta(x^*) \cap X$ and for all $y \in Y$ where β_δ is neighbouring around x^* .

However it is adapted if we optimize the function not on all of polyhedron X and Y but on the extreme points of those. The algorithm is presented below:

5.2.1 Phase I:

Find an extreme point x^1 admissible on X

Solve $\min\left\{f(x^1, y) / y \in Y\right\}$ to obtain an optimal y^1

Solve $\min\left\{f(x, y^1 / x \in X\right\}$ to obtain an optimal x^2

Repeat the procedure until to find a minimum local (x^*, y^*) such that $f(x^*, y^*) \leq f(x, y)$

5.2.2 PHASE II:

After determining the minimum local (x^*, y^*) we shall use the second approach which consists to globalize the alternative choice by sweeping all practicable space. We propose the following algorithm.

1. Let the local minimum x^*, y^* and $k = 2$
2. Let x^2,......., x^m the extreme points neighbouring to x^*
3. Solve $\min\left\{f(x^k, y) / y \in Y\right\}$ to obtain the solutions $y^k, k = 2, \ldots, m$
4. If $f(x^* y^*) \leq f(x^k y^k)$ for all k then stop with x^*, y^* a global minimum else choose $f(x^k, y^k) \leq f(x^*, y^*)$ and go to step 3 with $k = k + 1$.

Numerical Example

As in the linear case, it can be seen that if the constraint polyhedron is bounded then the convergence occurs after a finite number of iterations, with eventually a tie breaking rule in the degenerate case. It is important to note that if x^* and y^* are respectively optimal solutions for the two following problems:

$$\min\left\{f(x, y^*) : x \in X\right\} \tag{1}$$

$$\min\left\{f(x^*, y\ \ ) : x \in Y\right\} \tag{2}$$

Then (x^*, y^*) is not necessarily a solution of the problem. Indeed consider for instance the following example:

$$\min\ x_1 - x_2 - y_1 + (x_1, x_2)\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$s.t.\begin{bmatrix}1 & 4\\4 & 1\\3 & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\leq\begin{bmatrix}8\\12\\12\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}2 & 1\\1 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}\leq\begin{bmatrix}8\\8\\5\end{bmatrix}$$

$$x\geq 0 \qquad\qquad y\geq 0$$

If $y^*=(0,4)$, then , the function depending on x is $F(x,y^*)=5(x_1-x_2)$ and it is easily seen that the minimum for $x\in X$ is obtained at the point $x^*=(0,2)$ with the value -10.

Similarly $F(x^*,y)=y_1-2y_2-2$ and y^* gives the minimum on Y. Thus x^* and y^* solve (1) and (2), however $F(x^*,y^*)=-10$ whereas the global minimum value is -15 for x^* = (0, 3) and y^* = (0, 4).

We can extend and present some numerical examples to calculate the total cost of assignment for different problems and showing the implications of the algorithm.

Problem	Constraints	Optimal solution	Minimum cost
$Z=2x_1y_1+3x_1y_2+4x_2y_1+5x_2y_2$	$x_1+x_2=1$ $y_1+y_2=1$ $x_i;y_i\geq 0$	$x_1=1/3, x_2=2/3$ $y_1=1/3, y_2=2/3$ represented by x^*y^*	$Z=10/3$
$\sum_i\sum_j C_{ij}Z_{ij}$	$\sum_j Z_{ij}=1$ $\sum_i Z_{ij}=1$ $Z_{ij}\leq 1$	$x_1=0; x_2=1$ $y_1;y_2=0$ represented by x^*y^*	$Z=2$
3 jobs J1,J2,J3: represented by (2,3,4) 3 tasks T1,T2,T3 represented by (1,2,3) (Matrix form)	J1 to T1 with cost 2 J2 to T2 with cost 6 J3 to T3 with cost 12		Z=2+6+12=20

The bilinear assignment algorithm is a mathematical optimization technique used to assign asks to resources in the most efficient and optimal way possible. This algorithm has several implications, including:

Improved Efficiency: Tasks can be assigned to resources in the most optimal way possible, leading to improve defficiency in the overall process. This algorithm can help in minimizing the total time and cost of completing a project.

Increased Accuracy: The algorithm is designed to minimize the total distance or cost between tasks and resources. As a result, the allocation process becomes more accurate, leading to a better allocation of resources.

Better Resource Utilization: The algorithm can help in identifying the optimal utilization of resources by allocating the right task to the right resource. This algorithm can be used in various industries such as transportation, logistics, and manufacturing.

Sensitivity to Input Data: The algorithm is sensitive to the input data and the parameters used in the optimization process. Thus, the quality of the results may depend on the accuracy and completeness of the input data.

In summary, the bilinear assignment algorithm can lead to improve defficiency, increased accuracy, better resource utilization, and scalability.

This can also have implications for resource planning and decision-making. By understanding the optimal assignment of tasks to resources, decision-makers can better plan for future resource needs and allocate resources in a way that maximizes efficiency and productivity.

CONCLUSION:

Several applications of bilinear programs and solutions techniques for these problems have been reported in the literature, this paper describes an approach to search an optimal solution with the both techniques used here. We search for a local solution when it is possible to be found, and then we can find a global solution. A deep study is necessary to establish a characteristic of the solution, a rich bibliography in the book Global optimization of Horst Tuy states characteristics of the convex and concave function to determine the local minimum which becomes global, many applications can be modelled with this form and algorithms are encouraging.

In our opinion the new state of solution based on extreme points allows us to contribute to solving this problem. The properties and solutions of the constructed problems can be controlled by the users

The computational steps of the method are very easy in comparison to the other methods, which saves our time. Comparison of the proposed method with other methods for numerical examples shows that the proposed method gives identical results with those obtained by the other methods. This proves the validity of the method.

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