

## EXISTENCE AND UNIQUENESS OF THE STRONG SOLUTION FOR AN NON-LOCAL BOUNDARY VALUE PROBLEM

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### Abstract

In this work we are interested in a certain class of equations with operator coefficients and non-local boundary conditions. For the boundary value problem generated by these equations we establish the existence and uniqueness of the strong solution as well as its continuous dependence on the data. We use the method of energy inequality.

**Keywords:** Operator differential equation, Non-local boundary conditions, Energy inequality.

### Résumé

Dans ce travail, on s'intéresse à une certaine classe d'équations à coefficients opérationnelles et des conditions aux limites non locales. Pour le problème aux limites engendré par ces équations, on établit l'existence et l'unicité de la solution forte, ainsi que la dépendance continue par rapport aux données initiales. La méthode utilisée est celle des inégalités énergétiques.

**Mots clés:** Equation différentielle opérationnelle, Conditions aux limites non-locales, Inégalités énergétiques.

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This paper concerns the existence and uniqueness of a strong solution to a boundary value problem. More precisely, we search a function  $u$  solution of the problem:

$$\begin{aligned} \mathcal{L}u &= \frac{\partial^2 u}{\partial t_1 \partial t_2} + B \left[ \text{sign}(1 - |\mu_1|^2) \frac{\partial u}{\partial t_1} + \text{sign}(1 - |\mu_2|^2) \frac{\partial u}{\partial t_2} \right] \\ &+ \text{sign} \left[ (1 - |\mu_1|^2)(1 - |\mu_2|^2) \right] Au = f(t) \end{aligned} \quad (1)$$

$$\begin{aligned} \ell_{\mu_1} u &= u|_{t_1=0} - \mu_1 u|_{t_1=T_1} = \varphi(t_2); \\ \ell_{\mu_2} u &= u|_{t_2=0} - \mu_2 u|_{t_2=T_2} = \psi(t_1) \end{aligned}$$

where  $\mu = (\mu_1, \mu_2)$  in  $\mathbb{C}^2$ .

For  $\mu=0$ , we find the Goursat problem; this type of problem has been studied in [1, 2]. See also the recent results in [3-5]. The present work is an extension of the works [6] and [7], in two directions: first, the studied equation contains operator coefficients, and second, the variable  $t$  belongs to a bounded domain in  $\mathbb{R}^2$ . We proceed as follows:

We considered the problem as a resolution of the operator equation  $L_\mu u = F$ , where the operator  $L_\mu$  is considered from the Hilbert space  $E_\mu$  into the Hilbert space  $E$ , which will be defined later. For this operator, we establish an a priori estimate, then we prove the density of the range of this operator in the space  $E$ .

### POSITION OF THE PROBLEM

Let  $H$  be a Hilbert space with norm and inner product respectively denoted by  $|\cdot|$  and  $(\cdot, \cdot)$ . We suppose in the domain  $D = ]0, T_1[ \times ]0, T_2[ \in \mathbb{R}^2$  the differential equation:

$$\begin{aligned} \mathcal{L}u &= \frac{\partial^2 u}{\partial t_1 \partial t_2} + B \left[ \text{sign}(1 - |\mu_1|^2) \frac{\partial u}{\partial t_1} + \text{sign}(1 - |\mu_2|^2) \frac{\partial u}{\partial t_2} \right] \\ &+ \text{sign} \left[ (1 - |\mu_1|^2)(1 - |\mu_2|^2) \right] Au = f(t) \end{aligned} \quad (2)$$

with the non-local boundary data:

### ملخص

نهتم في هذا البحث بدراسة بعض أصناف المعدلات ذات عوامل مؤثرة و شروطها الحدية غير محلية. نبرهن على وجود و وحدانية الحل القوي للمسألة الحدية المولدة بهذه المعدلات التفاضلية، وكذلك الارتباط المستمر بالشروط الحدية. تعتمد البراهين على المترجمات الطا قوية. **الكلمات مفتاحية:** معادلات تفاضلية مؤثرة. شروط حدية غير محلية. المترجمات الطا قوية.

$$\begin{aligned} \ell_{\mu_1} u &= u|_{t_1=0} - \mu_1 u|_{t_1=T_1} = \varphi(t_2); \\ \ell_{\mu_2} u &= u|_{t_2=0} - \mu_2 u|_{t_2=T_2} = \psi(t_1) \end{aligned} \quad (3)$$

the functions  $u$  and  $f$  take values in  $H$ , the operators  $A$  and  $B$  are linear in  $H$ , with domains  $D(A)$  and  $D(B)$  everywhere denses in  $H$ . The parameter  $\mu = (\mu_1, \mu_2) \in \mathbb{C}^2$  and satisfies  $|\mu_i| \neq 1$ ,  $i=1,2$ . For study this problem, we impose the following conditions:

**C1:** The operator  $A$  is self-adjoint and satisfies:

$$(Av, v) \geq c_0 |v|^2, \quad \forall v \in D(A).$$

where  $c_0$  is a positive constant not depending on  $v$ .

**C2:** The operator  $B$  commutes with  $A$  and we have:

$$Re(Bv, v) > 0, \quad \forall v \in D(B).$$

**C3:** the functions  $\varphi$  and  $\psi$  take values in  $H$  and satisfy the compatibility conditions:

$$\varphi(0) - \mu_2 \varphi(T_2) = \varphi(0) - \mu_1 \psi(T_1).$$

To study this problem, we introduce the following spaces. We consider on  $D(A)$  the hermitian norm  $|u|_1 = |Au|$ .  $(D(A); | \cdot |_1)$  becomes a Hilbert space which we denoted by  $W^1$ , we then prove that  $A \in L(W^1, H)$ .

In a similar way, we construct the Hilbert space  $W^{1/2}$  on  $D(A^{1/2})$  provided with the norm  $|u|_{1/2} = |A^{1/2}u|$  and we show that  $A^{1/2} \in L(W^{1/2}, H)$ .

We denoted by  $H^1([0, T_i], W^{1/2})$  the completion of  $C^\infty([0, T_i], W^{1/2})$  with respect to the norms:

$$\|\varphi\|_1^2 = \int_0^{T_2} (|\varphi|^2 + |\varphi|_{1/2}^2) dt_2$$

$$\|\psi\|_1^2 = \int_0^{T_1} (|\psi|^2 + |\psi|_{1/2}^2) dt_1$$

Let  $H^{1,1}(\bar{D}, W)$  be the completion of the space  $C^\infty(\bar{D}, W)$  with respect to the norm:

$$\|u\|_{1,1}^2 = \int_D \left[ \left| \frac{\partial^2 u}{\partial t_1 \partial t_2} \right|_1^2 + \left| \frac{\partial u}{\partial t_1} \right|_1^2 + \left| \frac{\partial u}{\partial t_2} \right|_1^2 + |u|_1^2 \right] dt.$$

$E_\mu$  designs the completion of the space  $C^\infty(\bar{D}, W)$  with respect to the norm:

$$\|u\|_1^2 = \left[ \frac{\min(|1 - |\mu_1|^2|, |1 - |\mu_2|^2|)}{(1 + |\mu_1|^2)(1 + |\mu_2|^2)} \right]^2 \sup_{\tau \in D} ( \|u(\tau_1, \cdot)\|_1^2 + \|u(\cdot, \tau_2)\|_1^2 )$$

Let  $E$  be the Hilbert space  $L_2(D, H) \times \hat{H}^1((0, T_2)W^{1/2}) \times \hat{H}^1((0, T_1)W^{1/2})$  whose elements  $F = (f, \varphi, \psi)$  are such that

$$\|F\|_1^2 = \|f\|_1^2 + \|\varphi\|_1^2 + \|\psi\|_1^2 \quad \text{is finite}$$

where  $\hat{H}^1((0, T_2)W^{1/2}) \times \hat{H}^1((0, T_1)W^{1/2})$  is the closed subspace of the Hilbert space  $\hat{H}^1((0, T_2)W^{1/2}) \times \hat{H}^1((0, T_1)W^{1/2})$  whose elements  $\varphi$  and  $\psi$  are such that the condition **C3** is satisfies. We associate to problem (2)-(3) the operator  $L_\mu = (L, \ell_{\mu_1}, \ell_{\mu_2})$  with domain of definition:  $D(L_\mu) = H^{1,1}(\bar{D}, W)$ .

**Theorem 1.** Assume conditions **C1** and **C2** hold, then we have

$$\|u\|_1^2 \leq C \|L_\mu u\|_1^2 \quad \text{for every } u \in D(L_\mu) \quad (4)$$

where  $C$  is a positive constant not depending on  $u$  and  $\mu$ .

**Proof.** Multiply in  $H$ , the expression  $Lu$  by

$$Mu = \text{sign}(1 - |\mu_1|) \frac{\partial u}{\partial t_1} + \text{sign}(1 - |\mu_2|) \frac{\partial u}{\partial t_2}.$$

setting:

$$F_i(t_1, t_2) = \text{sign}(1 - |\mu_i|^2) \left[ \left| \frac{\partial u}{\partial t_i} \right|^2 + |u|_{1/2}^2 \right], \quad i=1,2$$

this yields:

$$2\text{Re}(Lu, Mu) - 2\text{Re}(B Mu, Mu) = \frac{\partial}{\partial t_2} F_1(t_1, t_2) + \frac{\partial}{\partial t_1} F_2(t_1, t_2) \quad (5)$$

We integrate the equality (5) with respect to  $(t_1, t_2)$  over the domain  $(0, \tau_1) \times (0, \tau_2)$ ; we get:

$$\begin{aligned} \int_0^{\tau_1} F_1(t_1, \tau_2) dt_1 + \int_0^{\tau_2} F_2(\tau_1, t_2) dt_2 = \\ 2\text{Re} \int_0^{\tau_1} \int_0^{\tau_2} ((Lu, Mu) - (B Mu, Mu)) dt \\ + \int_0^{\tau_1} F_1(t_1, 0) dt_1 + \int_0^{\tau_2} F_2(0, t_2) dt_2 \quad (\text{Eq1}) \end{aligned}$$

Similarly, integrating the equality (5) over respectively the domains:

$$(\tau_1, T_1) \times (\tau_2, T_2), (0, \tau_1) \times (\tau_2, T_2) \text{ and } (\tau_1, T_1) \times (0, \tau_2)$$

We then get the equalities (Eq 2), (Eq 3) and (Eq 4), which we sum up as follows:

$$\begin{aligned} (\text{Eq 1}) + \left[ \frac{1}{4} (1 + |\mu_1|^2) (1 + |\mu_2|^2) \right] (\text{Eq 2}) + \\ \frac{1}{2} (1 + |\mu_2|^2) (\text{Eq 3}) + \frac{1}{2} (1 + |\mu_1|^2) (\text{Eq 4}) \end{aligned}$$

We find:

$$\begin{aligned} \frac{1}{2} (1 - |\mu_1|^2) \left[ \int_0^{\tau_2} F_2(\tau_1, t_2) dt_2 + \frac{1}{2} (1 + |\mu_2|^2) \int_{\tau_2}^{T_2} F_2(\tau_1, t_2) dt_2 \right] + \\ \frac{1}{2} (1 - |\mu_2|^2) \left[ \int_0^{\tau_1} F_1(t_1, \tau_2) dt_1 + \frac{1}{2} (1 + |\mu_1|^2) \int_{\tau_1}^{T_1} F_1(t_1, \tau_2) dt_1 \right] = \\ \int_0^{\tau_2} \left[ F_2(0, t_2) - \frac{1}{2} (1 + |\mu_1|^2) F_2(T_1, t_2) \right] dt_2 + \\ \frac{1}{2} (1 + |\mu_2|^2) \int_{\tau_2}^{T_2} \left[ F_2(0, t_2) - \frac{1}{2} (1 + |\mu_1|^2) F_2(T_1, t_2) \right] dt_2 + \\ \int_0^{\tau_1} \left[ F_1(t_1, 0) - \frac{1}{2} (1 + |\mu_2|^2) F_1(t_1, T_2) \right] dt_1 + \\ \frac{1}{2} (1 + |\mu_1|^2) \int_{\tau_1}^{T_1} \left[ F_1(t_1, 0) - \frac{1}{2} (1 + |\mu_2|^2) F_1(t_1, T_2) \right] dt_1 + \\ 2\text{Re} \int_0^{\tau_1} \int_0^{\tau_2} ((Lu, Mu) - (B Mu, Mu)) dt_1 dt_2 + \frac{1}{4} (1 + |\mu_1|^2) (1 + |\mu_2|^2) \times \\ 2\text{Re} \int_{\tau_1}^{T_1} \int_{\tau_2}^{T_2} ((Lu, Mu) - (B Mu, Mu)) dt_1 dt_2 + \frac{1}{2} (1 + |\mu_2|^2) \times \end{aligned}$$

$$\begin{aligned}
 & 2 \operatorname{Re} \int_0^{\tau_1} \int_0^{\tau_2} ((Lu, Mu) - (BMu, Mu)) dt_1 dt_2 + \frac{1}{2} (1 + |\mu_1|^2) \times \\
 & 2 \operatorname{Re} \int_{\tau_1}^{\tau_2} \int_0^{\tau_2} ((Lu, Mu) - (BMu, Mu)) dt_1 dt_2 \quad (6)
 \end{aligned}$$

In order to majorize the right hand side of the equality (6), we need the following lemma:

**Lemma 1.** Let  $g$  be an arbitrary function  $g: D \rightarrow H$ . We defined by  $h$  the function  $h = g|_{t_i=0} - \mu_j g|_{t_i=T_i}$ ,  $i=1,2$ ;

$j=1,2$ , where  $\mu = (\mu_1, \mu_2)$  is a complex parameter satisfies  $|\mu_j| \neq 1$ . So, we have:

(i) if  $|\mu_j| < 1$ , then:

$$\left| g|_{t_i=0} \right|^2 - \frac{1}{2} (1 + |\mu_j|^2) \left| g|_{t_i=T_i} \right|^2 \leq \frac{(1 + |\mu_j|^2)}{(1 - |\mu_j|^2)} |h|^2.$$

(ii) if  $|\mu| > 1$ , then:

$$\left| g|_{t_i=0} \right|^2 + \frac{1}{2} (1 + |\mu_j|^2) \left| g|_{t_i=T_i} \right|^2 \leq \frac{(1 + |\mu_j|^2)}{(|\mu_j|^2 - 1)} |h|^2.$$

For the proof see [6].

Now, we use conditions  $C_2$ , we then divided the obtained inequality by  $(1 + |\mu_1|^2)(1 + |\mu_2|^2)$ , we find:

$$\begin{aligned}
 & \frac{\min \left[ |1 - |\mu_1|^2|, |1 - |\mu_2|^2| \right]}{4(1 + |\mu_1|^2)(1 + |\mu_2|^2)} \left[ \int_0^{\tau_2} \left( \left| \frac{\partial u}{\partial t_2} \right|^2 + |u|_{1/2}^2 \right) dt_2 + \right. \\
 & \left. \int_0^{\tau_1} \left( \left| \frac{\partial u}{\partial t_1} \right|^2 + |u|_{1/2}^2 \right) dt_1 \right] \leq (k_1^{-1} + k_2^{-1}) \int_0^{\tau_1} \int_0^{\tau_2} |Lu|^2 dt_1 dt_2 + \\
 & k_1 \int_0^{\tau_1} \int_0^{\tau_2} \left| \frac{\partial u}{\partial t_1} \right|^2 dt_1 dt_2 + k_2 \int_0^{\tau_1} \int_0^{\tau_2} \left| \frac{\partial u}{\partial t_2} \right|^2 dt_1 dt_2 + \\
 & \frac{\| \ell_{\mu_1} u \|_1^2 + \| \ell_{\mu_2} u \|_1^2}{\min \left[ |1 - |\mu_1|^2|, |1 - |\mu_2|^2| \right]} \quad (7)
 \end{aligned}$$

$$\text{where: } k_i = \frac{\min \left[ |1 - |\mu_1|^2|, |1 - |\mu_2|^2| \right]}{8(1 + |\mu_1|^2)(1 + |\mu_2|^2)T_{3-i}}, \quad i=1,2$$

Multiply (7) by  $\frac{\min \left[ |1 - |\mu_1|^2|, |1 - |\mu_2|^2| \right]}{4(1 + |\mu_1|^2)(1 + |\mu_2|^2)}$ , then by using some

other elementary estimates, we find:

$$\begin{aligned}
 & \frac{\min \left[ |1 - |\mu_1|^2|, |1 - |\mu_2|^2| \right]}{4(1 + |\mu_1|^2)(1 + |\mu_2|^2)} \left( \int_0^{\tau_2} \left( \left| \frac{\partial u}{\partial t_2} \right|^2 + |u|_{1/2}^2 \right) dt_2 + \right. \\
 & \left. \int_0^{\tau_1} \left( \left| \frac{\partial u}{\partial t_1} \right|^2 + |u|_{1/2}^2 \right) dt_1 \right) \leq \\
 & 32(T_1 + T_2) \int_0^{\tau_1} \int_0^{\tau_2} |Lu|^2 dt_1 dt_2 + 8 \left( \| \ell_{\mu_1} u \|_1^2 + \| \ell_{\mu_2} u \|_1^2 \right)
 \end{aligned}$$

Since the left hand side is not depending on  $\tau$ , we take the supremum on  $\tau \in D$ , we find the inequality (4), this achieves the proof of the theorem.

**Proposition 1.** The operator  $L_\mu$  admits a closure denoted  $\overline{L_\mu}$ , with domain of definition  $D(\overline{L_\mu}) = \overline{D(L_\mu)}$ .

**Proof.** See [4]

**Definition 1.** The solution of the equation  $\overline{L_\mu} u = F$ , where  $F = (f, \varphi, \psi)$ , is called a strong solution of problem (2)-(3).

By taking the limit we extend inequality (4) to strong

$$\text{solutions: } \|u\|_1^2 \leq \| \overline{L_\mu} u \|_1^2, \quad \forall u \in D(\overline{L_\mu})$$

and we have:  $R(\overline{L_\mu}) = \overline{R(L_\mu)}$  and  $(\overline{L_\mu})^{-1} = \overline{(L_\mu^{-1})}$ .

From the inequality (8), we deduce the uniqueness of strong solution and the closure of  $R(\overline{L_\mu})$ . For the existence of the strong solution, it remains to prove that the range  $R(L_\mu)$  is dense in  $E$ , which is equivalent to  $R(L_\mu)^\perp = \{(0,0,0)\}$ .

## EXISTENCE OF THE STRONG SOLUTION

Let  $V = (v, \varphi_1, \psi_1) \in R(L_\mu)^\perp$ , then for  $u \in D(L_\mu)$  we have:  $(L_\mu u, V)_E = (Lu, v)_{L_2(D,H)} + (\ell_{\mu_1} u, \varphi_1)_1 + (\ell_{\mu_2} u, \psi_1)_1 = 0$  (9)

Choosing  $u$  such that  $\ell_{\mu_1} u = \ell_{\mu_2} u$ , the equality (9) becomes:

$$(L_\mu u, V)_E = (Lu, v)_{L_2(D,H)} = \int_D (Lu, v) dt = 0 \quad (10)$$

Let  $\omega$  be the solution of the problem:

$$\frac{\partial^2 \omega}{\partial t_1 \partial t_2} = v; \quad \overline{\mu_1} \omega \Big|_{t_1=0} - \omega \Big|_{t_1=T_1} = 0, \quad \overline{\mu_2} \omega \Big|_{t_2=0} - \omega \Big|_{t_2=T_2} = 0 \quad (11)$$

$$\text{Setting: } \omega = Ag \quad \text{then} \quad v = A \frac{\partial^2 g}{\partial t_1 \partial t_2}$$

Replacing  $v$  by this expression in (10), integrating by part and using (11) this yields:

$$\iint_D \left( Lu, A \frac{\partial^2 g}{\partial t_1 \partial t_2} \right) dt = \iint_D \left( A \frac{\partial^2 u}{\partial t_1 \partial t_2}, L'g \right) dt = 0 \quad (12)$$

where:

$$\begin{aligned}
 L'g &= \frac{\partial^2 g}{\partial t_1 \partial t_2} - B^* \left[ \operatorname{sign}(1 - |\mu_1|^2) \frac{\partial g}{\partial t_1} + \operatorname{sign}(1 - |\mu_2|^2) \frac{\partial g}{\partial t_2} \right] \\
 &+ \operatorname{sign} \left[ (1 - |\mu_1|^2)(1 - |\mu_2|^2) \right] Ag
 \end{aligned}$$

We denoted by  $L'_\mu$  the differential operator generated by the problem:

$$\left\{ \begin{aligned}
 L'g &= \frac{\partial^2 g}{\partial t_1 \partial t_2} - B^* \left[ \operatorname{sign}(1 - |\mu_1|^2) \frac{\partial g}{\partial t_1} + \operatorname{sign}(1 - |\mu_2|^2) \frac{\partial g}{\partial t_2} \right] \\
 &+ \operatorname{sign} \left[ (1 - |\mu_1|^2)(1 - |\mu_2|^2) \right] Ag \\
 \ell'_{\mu_1} g &= \overline{\mu_1} g \Big|_{t_1=0} - g \Big|_{t_1=T_1} = 0; \\
 \ell'_{\mu_2} g &= \overline{\mu_2} g \Big|_{t_2=0} - g \Big|_{t_2=T_2} = 0
 \end{aligned} \right.$$

Since the set  $\left\{A \frac{\partial^2 u}{\partial t_1 \partial t_2}\right\}$  is dense in  $L_2(D, H)$ , from the (12), we deduce that  $L'g = 0$ . By techniques similar to those used to prove the estimate (4), we prove that:

$$\|g\|_1^2 \leq C' \|L'_{\mu} g\|_1^2$$

So, since  $L'_{\mu} g = 0$ , we conclude that  $g = 0$ , then  $v = 0$ . The equality (9) becomes:

$$(\ell_{\mu_1} u; \varphi_1)_1 + (\ell_{\mu_2} u; \psi_1)_1 = 0, \quad \forall u \in D(L_{\mu})$$

As the domain of the operator  $(\ell_{\mu_1}, \ell_{\mu_2})$  is dense in the product space  $\hat{H}^1((0, T_2), W^{1/2}) \times \hat{H}^1((0, T_1), W^{1/2})$  then  $\varphi_1 = \psi_1 = 0$  and consequently  $R(L_{\mu})$  is dense in  $E$ .

**Theorem 4.** Under conditions of theorem 1, the conditions  $C_1$  and  $C_2$ , we have for all  $f \in L_2(D, H)$ ,  $\varphi \in \hat{H}^1((0, T_2), W^{1/2})$ ,  $\psi \in \hat{H}^1((0, T_1), W^{1/2})$ , satisfying  $\varphi(0) - \mu_2 \varphi(T_2) = \psi(0) - \mu_1 \psi(T_1)$ , there exists one and only one solution  $u \in E_{\mu}$  for the problem (2)-(3) and the following inequality holds:

$$\|u\|_1^2 \leq C(\|f\|^2 + \|\varphi\|_1^2 + \|\psi\|_1^2)$$

where  $C$  is not depending on  $u, f, \varphi, \psi, \mu_1$  and  $\mu_2$ .

#### CONTINUITY OF THE SOLUTION WITH RESPECT TO THE PARAMETERS

Let  $E^1$  be the completion of the space  $C^{\infty}(D, W^1)$  with respect to the norm:

$$\|u\|_{E^1}^2 = \sup_{\tau \in D} \left[ \int_0^{T_1} \left( \left| \frac{\partial u}{\partial t_1} \right|^2 + |u|_{1/2}^2 \right) dt_1 + \int_0^{T_2} \left( \left| \frac{\partial u}{\partial t_2} \right|^2 + |u|_{1/2}^2 \right) dt_2 \right]$$

**Theorem 5.** If the conditions of theorem 4 hold, then for the convergent sequence

$$(\mu_{1n}, \mu_{2n}) \xrightarrow{n \rightarrow \infty} (\dot{\mu}_1, \dot{\mu}_2); \quad |\dot{\mu}_i| \neq 1 \quad i = 1, 2$$

we have:

$$(\bar{L}_{\mu_n})^{-1} \xrightarrow{n \rightarrow \infty} (\bar{L}_{\dot{\mu}})^{-1}$$

in  $L(E, E^1)$  provided with the topology of simple convergence.

**Proof.** Since the constant in the inequality (8) is not depending on  $\mu$  and the norm in the space  $E_{\mu_n}$  is minored by the norm of  $E^1$  with a constant not depending on  $\mu_n$ , then we get:

$$\|u\|_{E^1}^2 \leq C \|\bar{L}_{\mu_n} u\|_1^2 \quad \forall u \in D(\bar{L}_{\mu_n}) \quad (13)$$

where  $C$  is not depending on  $u$  and  $\mu_n$ .

From this, we deduce that  $\sup \left\| (\bar{L}_{\mu_n})^{-1} \right\|_{L(E, E^1)}$  is finite.

Since  $R(L_{\dot{\mu}})$  is dense in  $E$ , it suffices to establish that:

$$(\bar{L}_{\mu_n})^{-1} \xrightarrow{n \rightarrow \infty} (\bar{L}_{\dot{\mu}})^{-1} \text{ in } R(L_{\dot{\mu}}).$$

For all  $F \in R(L_{\dot{\mu}})$ , we have:

$$(\bar{L}_{\mu_n})^{-1} F - (\bar{L}_{\dot{\mu}})^{-1} F \in D(\bar{L}_{\mu_n})$$

From (13), we deduce that:

$$\left\| (\bar{L}_{\mu_n})^{-1} F - (\bar{L}_{\dot{\mu}})^{-1} F \right\|_{E^1}^2 \leq C \left\| F - L_{\mu_n} (\bar{L}_{\dot{\mu}})^{-1} F \right\|_{E^1}^2 \quad (14)$$

and if  $(L_{\dot{\mu}})^{-1} F = h$ , then for sufficiently great  $n$  and for  $h \in H^{1,1}(D, W)$ , we have:

$$\begin{aligned} \left\| L_{\dot{\mu}} h - L_{\mu_n} h \right\|^2 &= |\dot{\mu}_1 - \mu_{1n}|^2 \int_0^{T_1} \left( \left| \frac{\partial h}{\partial t_1} \right|^2 + |h|_{1/2}^2 \right) dt_1 + \\ &+ |\dot{\mu}_2 - \mu_{2n}|^2 \int_0^{T_2} \left( \left| \frac{\partial h}{\partial t_2} \right|^2 + |h|_{1/2}^2 \right) dt_2 \end{aligned}$$

This, together with the inequality (14), implies:

$$\left\| (\bar{L}_{\mu_n})^{-1} F - (\bar{L}_{\dot{\mu}})^{-1} F \right\|_{E^1}^2 \rightarrow 0 \text{ when } (\mu_{1n}, \mu_{2n}) \xrightarrow{n \rightarrow \infty} (\dot{\mu}_1, \dot{\mu}_2)$$

for each  $F \in R(L_{\dot{\mu}})$ . This completes the proof.

#### REFERENCES

- [1]- N. Brich N.I., Yurchuk N.I., "Goursat problem for a second order abstract linear differential equation". *Differ. Uravn.*, T.7, N° 6, (1971), pp. 1017-1030 (Russian).
- [2]- Guezane-Lakoud A., Rebbani F., Yurchuk N.I., "Problème aux limites pour une équation différentielle opérationnelle de second ordre", *Maghreb. Mathematical Review*, Vol.6, N°1, (1997), pp. 39-48 (French).
- [3]- Guezane-Lakoud A., Rebbani F., "A priori estimate for hyperbolic abstract equation with non local boundary conditions", *Mathematical Maghreb Review*, Vol.8, N°1&2, (1999), pp.111-118.
- [4]- Guezane-Lakoud A., Rebbani F., "Strong solution for an abstract non local boundary value problem", *Int. J. of Applied Math.*, Vol. 4, N°4, (2000), pp. 469-478.
- [5]- Guezane-Lakoud A., "On a class of hyperbolic equation with abstract non-local boundary conditions", *Int. J. of Applied Sc. Computations*, Vol.8, N°1, (2001).
- [6]- Chesalyn V.I., Yurchuk N.I., "Problem with non-local boundary conditions for an Liav's abstract equation", *I. zv. Akad. Nauk. BSSR. Serie Phys. Mat.*, N° 6, (1973), pp. 30-35, (Russian).
- [7]- Engelman U., "Boundary value problem for an operator hyperbolic partial differential equation of second order with non-local boundary conditions", *Differ. Uravn.*, T.11, N°9, (1975), pp. 1687-1963 (Russian).  $\square$