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COMPUTATION OF MASS OSCILLATIONS IN A SURGE TANK BY FINITE ELEMENT TECHNIQUE

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ABSTRACT

The fluctuation of free surface water in surge tanks is of great interest for its design. For this, various mathematical models exist for the analysis of mass oscillations in a surge shaft following valve adjustment. This paper describes a finite element technique using weighted residuals method for the solution of the governing differential equations. Three weighting functions are applied and the results are compared with those from classical methods. First, a relatively simple case of surge analysis with a sudden load rejection in the penstock is analyzed but neglecting tunnel friction. Then friction is included for test and comparison. The results indicate that the proposed numerical approach leads to an accurate simulation of the water surface motion.

Keywords: mass oscillation, surge tank, finite element method, weighted residuals.

INTRODUCTION

The water surface motion in a surge tank following closure or opening of valves has received considerable attention since several years. Being an important hydraulic structure surge tank is an integral part of any hydroelectric scheme in which, owing to changes in load and running condition, the rate of flow of water in the supply pipeline may be required to undergo rapid changes. The behaviour of the water surface in a surge tank is therefore complex and is influenced by several factors. A complete and satisfactory solution of the governing differential equations which predict the height of the surges at any given instant of time is restricted to a few simple problems of instantaneous shutdown (Eydoux, 1917; Jaeger, 1954). For this, depending on the complexity of the problem and assumptions made, a variety of methods are available for the approximate surge analysis. These methods may be classified as analytical, analogue, graphical or numerical. Generally analytical solutions are only available for a few special cases while graphical methods (as Schoklitsch approach) have become obsolete and rarely used (Escande, 1950). The numerical methods in particular have generally received wide acceptance with the advent of computers with large accessibility. Hens it becomes now possible to obtain quick and fairly accurate numerical solutions much more economically than with any other method.

The numerical approaches have almost exclusively used some form of finite difference approximation of the basic equations and several schemes based on an iterative solution have been reported (Jaeger, 1977). Latter work concentrating mainly on developing refinements of existing methods (France, 1977; 1980). One of the powerful numerical approach used in engineering science and which has been extensively developed in fluid mechanics during the three past decade is that of Finite Element Method (Cannor and Brebbia,1976; Zienkiewicz and Taylor, 2000; Hervouet, 2007). In hydraulic transient field the FEM was successfully applied to simulate water hammer problems based on elastic theory model (Szymkiewicz and Mitosek, 2004). With presence of surge tank in hydraulic system where rigid column theory can by applied, McKeogh and France (1983) and France (1984) were first to apply finite element technique with satisfactory results for case of instantaneous valve operations. He applies weighting residual process to the second order differential equation and leads to a three points recurrence formula which is not auto starting method. Noting also that this approach leads to relatively complicate algebraic equations and some difficulties to trait case of slow valve operations.

In this paper we present a simple technique based on finite element method in time domain for the computation of water level oscillation in a surge tank. For this, partial discretization approach is used to solve the decoupled form of the ordinary differential equations governing the mass oscillation problems. The FEM is tested for various weighting residual functions with Runge-Kutta method well known for his accuracy in these typical problems.

BASIC EQUATIONS

The governing equations describing the mass oscillations of the whole body of water in the pipeline and surge tank is based upon three fundamental equations (i) the dynamic equation (ii) the equation of continuity and (iii) an equation giving the velocity of the water surface. (Jaeger, 1954; Escande, 1971; Bergeron, 1970; Featherstone and Nalluri, 1995):

(i) Dynamic equation:

$$\frac{L}{g}\frac{dw}{dt} + z \pm Rw^2 = 0 \tag{1}$$

(ii) Continuity equation:

$$wS = vA + Q \tag{2}$$

(iii) Water surface velocity:

$$v = \frac{dz}{dt} \tag{3}$$

Where L and S are respectively the length and the area of the supply tunnel, w the flow velocity in the tunnel, A the area of the surge tank, z the elevation of the water level in the surge tank relative to the reservoir water level, Q the discharge in the penstock to the turbines, R the friction factor (tunnel and throttle) and t the time. The (+) sign for the friction loss is taken when the flow is from the reservoir into the surge tank and the (-) sign is taken when the flow reverses.

Combining equations (1), (2) and (3) leads to:

$$\frac{LA}{gS}\frac{d^2z}{dt^2} + \frac{L}{gS}\frac{dQ}{dt} + z \pm \frac{RA^2}{S^2}\left(\frac{dz}{dt}\right)^2 \pm \frac{2RA}{S}\frac{dz}{dt} \pm \frac{RQ^2}{S^2} = 0 \tag{4}$$

Equation (4) is a nonlinear second order differential equation. Direct integration of equations (1) and (2) is only possible in a few special cases (Jaeger, 1977). If the friction losses in the tunnel are included, the only case where a direct solution of equation (4) is possible is that of total closure or sudden complete rejection of turbine load (Escande, 1971). For real practical problems involving various flow annulations time function $Q = Q(t)$ and multiple surge tank forms, the resolution of the mass oscillations mathematical model is only possible using numerical methods.

FINITE ELEMENT TECHNIQUE

Numerical approach using a finite element method in time was successfully used for resolution of unsteady problems. In this technique, called partial discretization (Zienkiewicz and Taylor, 2000), we use time approximation scheme to convert ordinary differential equations to algebraic equations. This is achieved by discretising time domain and applying the weighting residual process within each time increment leading to a recurrence formula (France, 1984). To solve the mass oscillations problem we will consider the decoupled equations (1), (2) and (3) which can be written in general form:

$$C\dot{a} + Ka + f = 0 \tag{5}$$

Were C and K are constant or variable dependent parameters and $\dot{a} = da/dt$. Discretising equation (5) into finite element of time of length Δt, the variable a approximated in terms of its nodal values is in the form (*Reddy, 1993*)

$$a(t) \approx \hat{a} = \sum_{i=1}^{n} a_i N_i \tag{6}$$

In which a_i are the unknown values at the nodes and N_i the shape (interpolation) function defined continuously within each element (Figure 1).

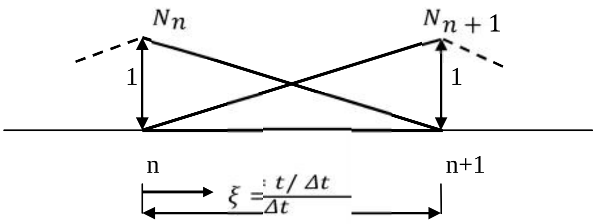

Figure 1: Shape function for two points recurrence formula

The objective is to obtain an approximation for a_{n+1} given the value of a_n and the forcing vector f acting in the interval of time Δt. The shape functions and their first time derivatives can be written in terms of local variables as:

$$
\begin{array}{ll}
0 \le \P \le 1 & \P = t / \Delta t \\
N_n = 1 - \P & \dot{N}_n = -t / \Delta t \\
N_{n+1} = \P & \dot{N}_{n+1} = t / \Delta t
\end{array} \tag{7}
$$

Substituting equation (6) into (5) and applying a typical weighted residual equation for an element of length Δt (Cannor and Brebbia, 1976; Zienkiewicz and Taylor, 2000) yields :

$$\int_0^1 \cdot \left[C\left(a_n \dot{N}_n + a_{n+1}\dot{N}_{n+1}\right) + K\left(a_n N_n + a_{n+1} N_{n+1}\right) + f \right] d\P = 0 \tag{8}$$

In which $\cdot$ is any weighting function. As the problem is an initial value one, a_n is assumed known (or can be determined from steady state conditions), then equation (8) will serve to determine a_{n+1} approximately. Substituting the interpolation function and his corresponding derivative into equation (8) gives the required recurrent formula (Zienkiewicz and Taylor, 2000):

$$\left(K\int_0^1 \cdot \ \P d\P + C\int_0^1 \cdot \ d\P / \Delta t \right) a_{n+1} + \left(K\int_0^1 \cdot \ (1 - \P) d\P - C\int_0^1 \cdot \ d\P / \Delta t \right) a_n + \int_0^1 \cdot \ f d\P = 0 \tag{9}$$

Introducing $\overline{\ }$ as a weighting parameter and $\bar{f}$ as average value of f given by:

$$\overline{\ } = \int_0^1 \cdot \ \P \, d\P \Big/ \cdot \ d\P \tag{10}$$

$$\bar{f} = \int_0^1 \cdot \ f \, d\P \Big/ \cdot \ d\P \tag{11}$$

We can write immediately:

$$\left(C / \Delta t + K \overline{\ } \right) a_{n+1} + \left(-C / \Delta t + K\left(1 - \overline{\ }\right)\right) a_n + \bar{f} = 0 \tag{12}$$

Hence a_{n+1} can be solved providing the variation of water level with time. To solve for a_{n+1} one initial conditions a_n have to be specified which is the initial water level in the surge tank at t = 0 and will therefore be equal to the friction head lost under steady state conditions. Many weighting functions can be inserted into expressions (10) to yield ¯ which is then substituted into the basic recurrence formula (12). Three weighting functions are considered, namely Galerkin method · $= N$ (¯ $= 2/3$), point collocation using the Dirac function · $=$ ¢$_i$ (¯ $= 1$ and ¯ $= 0$) and Subdomain collocation · $= 1$ (¯ $= 1/2$). Note that equation (1) is nonlinear hens the numerical approach using equation (5) is solved iteratively at each time step.

APPLICATION

The purpose of this section is to compare and evaluate the accuracy of the numerical model presented herein in solving the mass oscillation equations. The data of the hydraulic installation tested is taken from literature (Escande, 1950) which are as follow (table 1):

Table 1: Summary of the installation data

Area of the surge tank	300 m²
Tunnel cross section	10 m²
Length of the tunnel	4000 m
Initial steady flow	20 m^3/s
Steady state head losses	4.105 m

The numerical tests are carried out for a frictionless case and the case considering the friction losses in the tunnel pipe for an instantaneous load rejection i.e. a sudden valve closure.

Test I: Frictionless case

This test aims to compare the numerical results with the analytical solution. Neglecting head losses in the system the water level oscillations in the surge tank are sinusoidal. Results for this case when a time step $\Delta t = 2\ s$ is chosen are recapitulated in table 2.

Table 2: Comparison of results (test I)

Method	Amplitude (m)	Relative error (%)
Analytical	7.37335018	-
Point collocation (¯ $= 0$)	7.47894532	1.432
Point collocation (¯ $= 1$)	7.26950644	1.408
Subdomain collocation (¯ $= 1/2$)	7.37332905	$2.87\ 10^{-4}$
Galerkin (¯ $= 2/3$)	7.33845457	0.47
Runge-Kutta 4	7.37332829	$2.9\ 10^{-4}$
Escande	7.385919	0.171

It is thus noted that the precision of the solution obtained by the finite element method is related to the type of weighting function. The best solution is obtained by the subdomain collocation method where a relative error of 2.87x 10^{-4} is noted. It was shown by Zienkiewicz and Taylor (2000) the minimum of truncation error is obtained for ¯ $= 1/2$. This fact explains why this numerical approach (Crank-Nicolson scheme) possesses higher accuracy. For this case the Runge-Kutta method gives a similar numerical solution but for a higher computational effort.

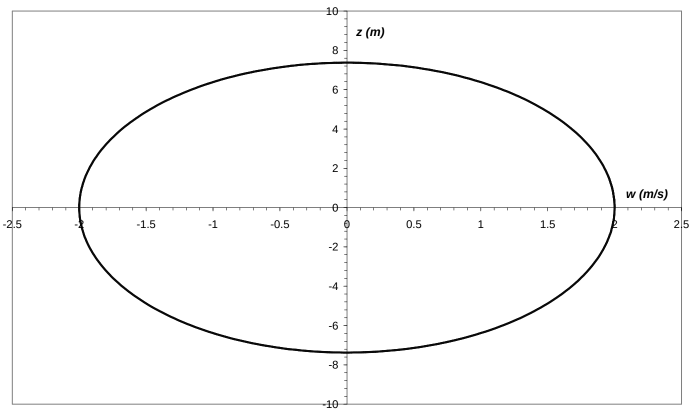

Figure 2: State space of results (test I)

Reporting results of the temporal history of the water surface fluctuation on a state space (w, z) one can see that the curve takes a closed ovoid form (Figure 2). The curve thus illustrates balance between the kinetic and potential energy at every moment without dissipation.

Test II: Frictional case

The introduction of the dissipative term into the computation makes the dynamic equation (1) nonlinear. Thus the use of the general equation model (5) imposes a linearization of equation (1). This is carried out while posing $K = R|w|$ in the numerical model and an iterative process is necessary at each time step to ensure convergence of the numerical computation.

The results of water level variation with respect to time in the surge tank are then reported on figure (3) and comparison between various numerical schemes is reported on table 3 for the first upsurge.

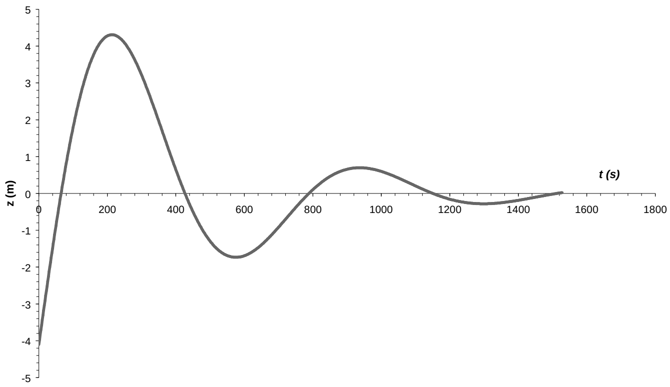

Figure 3: Temporal history of water level in the surge tank (test II)

We can note then for different weighting functions in finite element model the numerical results are very close. A comparison with the analytical solution available in this case for the maximum surge (Jaeger, 1977) shows that relative errors are so small and the numerical solution computed meaning finite element techniques is accurate. However it can be seen from table 3 that discrepancies noted for the numerical schemes are reduced in Galerkin's and subdomain's method. Because of the purely explicit character of the point collocation approach ($\overline{\ } = 0$), which correspond to the Euler method, the numerical scheme gives the least accuracy and presents a high sensitivity to the time step choice.

Table 3: Comparison of numerical solutions (test II)

Method	**Upsurge (m)**	**Relative error (%**
Analytical	4.92812573	-
Point collocation (¯ $= 0$)	5.02057558	1.88
Point collocation (¯ $= 1$)	4.86189238	1.34
Subdomain collocation (¯ $= 1/2$)	4.94046605	0.25
Galerkin (¯ $= 2/3$)	4.91410536	0.28
Runge-Kutta	4.92975339	0.03

Comparatively to the Runge-Kutta method the finite element technique (with first order recurrence formula) leads to a little less precision due to the fact that the first one uses a fractional step approach and the second one is to a single step. This can be achieved using a high order shape function leading to a multi-step method. The computed solution is reported on a state plan (figure 4).

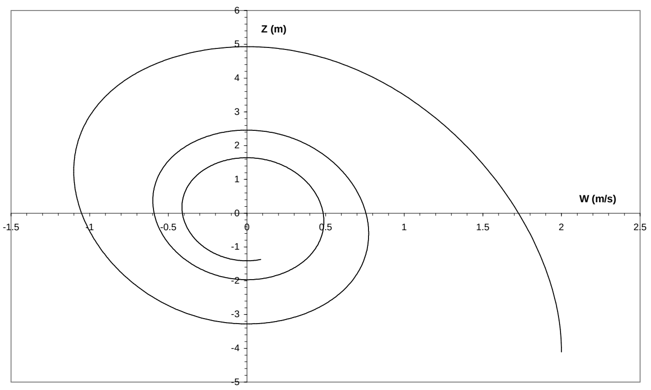

Figure 4: State space of results (test II)

Due to the physical dissipation for this case the solution curve presents a spiral form converging towards the center which corresponds to the static regime.

CONCLUSIONS

The complete analysis of hydraulic transients protection in pressurized flow is usually carried out using numerical solution of the classical water hammer partial differential equations. However, in certain cases when a surge tank is used in the hydraulic system the elastic effects can be dropped and the phenomenon is called mass oscillations. Due the assumption that the tunnel and the liquid are rigid the mathematical model describing the phenomenon are a set of nonlinear ordinary differential equations. Thus a closed-form solution is

available only for few practical cases and a numerical integration is required.
The finite element method using a weighted residual process has been presented for the analysis of mass oscillations in surge tanks. Various weighting functions, point collocation, subdomain collocation and Galerkin's method was tested and compared to other numerical and analytical solution. For linear shape function the technique of finite element leads to two points recurrence formula. Depending on the weighting parameter¯ , the numerical scheme can be explicit or implicit. For the case of frictionless flow it has been shown that finite element technique gives accurate results and is in well agreement with the Runge-Kutta method and the exact sinusoidal solution. In this case the subdomain collocation method leads to the best results. For the case of frictional flow a very small differences between weighting functions was noted in the evaluation of the maximum upsurge. However, for the two cases simulated the point collocation method presents the least accuracy compared to the others weighting functions.
The numerical approach presented herein is simple and accurate for the resolution of the nonlinear mass oscillation equations. A high precision computation can be achieved using a highest order shape function but requires more computation efforts.

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