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Semigroup's approach to the study of the Hölder continuous regularity for Laplace equation in nonsmooth domain.

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ABSTRACT

We will investigate the Dirichlet problem for Laplace equation set in an singular domain with cuspidal point. We look to describe the behavior of the unique solution near the cuspidal point in the framework of the little Hölder space $h^{2\sigma}(\Omega)$ with $\sigma \in]0, 1/2[$.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ a bounded domain. We assume that its boundary $\partial\Omega$ is of class C^∞ except at the origin $(0,0)$ where $\partial\Omega$ has a cuspidal point. To be more precise, assume that we can choose cartesian coordinates so that

$$\Omega_{x_0} := \Omega \cap B(0, x_0) = \left\{(x, y) \in \mathbb{R}^2 : 0 < x < x_0,\, 0 < y < \psi(x)\right\},$$

where $B(0, x_0)$ is the ball of center 0 and radius x_0. Here $x_0 > 0$ is small enough and ψ is a real function satsifying the following conditions

1. $\psi \in C^2([0, x_0]) \cap C^\infty(]0, x_0])$.
2. $\psi < 0$ on $]0, x_0]$.
3. $\int_0 \frac{dt}{\psi(t)}$ diverges.
4. $\psi(0) = \psi'(0) = 0$.

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5. $\psi(0)\psi''(0) = 0$.
6. We assume also that ψ can be extended to $[x_0, +\infty[$, so that $\dfrac{1}{\psi}$ remains in $L^1(]x_0, +\infty[)$.

Consider the problem

$$\Delta u = h, \quad \text{in } \Omega, \tag{1}$$

under the homogenous boundary conditions

$$u = 0, \quad \text{on } \partial\Omega. \tag{2}$$

The right hand term h is taken in the little Hölder space $h^{2\sigma}(\Omega)$ which denotes the subspace of $C^{2\sigma}(\Omega)$ consisting of the functions f such that

$$\forall (x,y), (x',y') \in \Omega : \lim_{\delta\to 0^+} \sup_{0<\|(x,y)-(x',y')\|\leq\delta} \frac{\|f(x,y) - f(x',y')\|_E}{\|(x,y)-(x',y')\|^{2\sigma}} = 0.$$

We assume also that

$$h(x_0, 0) = h(x_0, \psi(x_0)) = 0, \tag{3}$$

(more details about these spaces will be given later).
The study of elliptic problems posed in cusp domains was considered by numerous authors, see e.g. [7] and [12] (and the extensive bibliography therein). The majority of these works deals with the L^p setting of these problems. Comparatively, there are a few results concerning the little Hölder regularity. The difficulty related to this kind of problems comes from the typical properties of the functional framework. In our situation, the classical methods such as the variational method do not apply, see [9], [10]. Furthermore, we know that such spaces have not the UMD character, see [3]. This explains why the operatioanl approach used in [1], [6], [7] is excluded. Hence, we use an alternative approach, namely the theory of abstract differential equations. This technique has been fruitfully used in [4] to prove some regularity results for Problem (1). These results are restricted to the domain

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : 0 < x < x_0, \ -x^\alpha < y < x^\alpha\right\},$$

with $1 < \alpha \leq 2$.
In this work, we follow the same strategy. Our goal is to give a complete study of the problem (1) in the neighborhood of the cusp edge it means in Ω_{x_0}. We will prove that Problem (1)-(2) has a unique strict solution. Furthermore, we show that the regularity of this solution near the cuspidal edge is dependent to the geometry of the domain Ω_{x_0} and the exponent σ.

The schedule of the paper is the following one : In Section 2, we introduce some notations and definitions concerning some functions spaces to be used throughout this paper. Section 3, there are two main steps. First, we use an appropriate change of variables to transform our singular domain Ω into a fixed one. Secondly, we write the transformed problem as an abstract differential equation of elliptic type. Section 4, is devoted to the complete study of the abstract version of the transformed problem, the techniques are essentially based on the use of the semigroup's theory and some interpolation spaces. Section 5, we go back to the original domain and we give our main result describing the regularity of the solution u of Problem (1)-(2).

2. Preliminaries

In this work, it is necessary to introduce some Banach spaces of vector-valued functions. Let $(E, \|.\|_E)$ be a complex Banach space and let $\mu \in]0,1[$. We consider the following functional spaces $B([0,+\infty[;E), C([0,+\infty[;E), C^2([0,+\infty[;E)$ consisting respectively of the bounded , continuous, 2 times continuously differentiable functions $f : [0,+\infty[\to E$.

We set also

$$C_b([0,+\infty[;E) = \left\{ f \in C([0,+\infty[;E) : \lim_{\xi \to +\infty} f(\xi) = 0 \right\},$$

$$C_b^2([0,+\infty[;E) = C^2([0,+\infty[;E) \cap C_b([0,+\infty[;E).$$

The Banach spaces of Hölder continuous functions $C^\mu([0,+\infty[;E)$ is defined by

$$C_b^\mu([0,+\infty[;E) = \left\{ f \in C_b([0,+\infty[;E) : \sup_{\xi_1 > \xi_0 \geq 0} \frac{\|f(\xi_1) - f(\xi_0)\|_E}{|\xi_1 - \xi_0|^\mu} < +\infty \right\}, \text{with}$$

$$\|f\|_{C_b^\mu([0,+\infty[;E)} := \|f\|_{C([0,+\infty[;E)} + \sup_{\xi_1 > \xi_0 \geq 0} \frac{\|f(\xi_1) - f(\xi_0)\|_E}{|\xi_1 - \xi_0|^\mu}.$$

The Banach spaces of little Hölder continuous functions $h_b^\mu([0,+\infty[;E), h_b^{\mu+2}([0,+\infty[;E)$ are defined by

$$h_b^\mu([0,+\infty[;E) = \left\{ f \in C_b^\mu([0,+\infty[;E) : \lim_{\delta \to 0} \sup_{\xi_1 > \xi_0 \geq 0, |\xi_1 - \xi_0| \leq \delta} \frac{\|f(\xi_1) - f(\xi_0)\|_E}{(\xi_1 - \xi_0)^\mu} = 0 \right\}.$$

$$h_b^{\mu+2}([0,+\infty[;E) = \left\{ f \in C_b^2([0,+\infty[;E) : f, f', f'' \in h^\mu([0,+\infty[;E) \right\}.$$

$$L^\infty(]0,+\infty[;E) = \left\{ f :]0,+\infty[\to E, \text{ Bochner measurable and } \sup_{\xi \in U} ess \|f(\xi)\|_E < \infty \right\}.$$

Remark 1 *for $\mu \in]0,1[$, One has*

$$C_b^2([0,+\infty[,E) \subset C_b^1([0,+\infty[,E) \subset h_b^\mu([0,+\infty[,E) \subset C_b^\mu([0,+\infty[,E) \subset C_b([0,+\infty[,E).$$

Remark 2 *For $\mu \in]0,1[$, every function of $h^\mu(\Omega)$ can be extended to a function of $h^\mu(\overline{\Omega})$.*

3. Change of variables

We set

$$\begin{aligned} \Pi \quad & : \Omega_{x_0} \to Q_{\xi_0} \\ & (x,y) \mapsto \left(\xi := \theta^{-1}(x) := -\int_x^{+\infty} \frac{d\sigma}{\psi(\sigma)}, \eta := \frac{y}{\psi} \right), \end{aligned}$$

where Q_{ξ_0} is the semi-ifinite strip

$$Q_{\xi_0} =]\xi_0, +\infty] \times]0,1[, \ \xi_0 = -\int_{x_0}^{+\infty} \frac{d\nu}{\psi(\nu)},$$

which mean that the cuspidal point $(0,0)$ is transformed in

$$D_{\infty}=\{(+\infty,\eta):\eta\in\left]0,1\right[\}=\{+\infty\}\times\left]0,1\right[. \tag{4}$$

Now, consider the following change of functions

$$\begin{cases} v(\xi,\eta):=u(x,y), \\ \\ g(\xi,\eta):=h(x,y). \end{cases} \tag{5}$$

Consequently. Problem (1)-(2) is equivalent to

$$\begin{cases} \Delta_{(\xi,\eta)}v(\xi,\eta)+[Lv](\xi,\eta)=f(\xi,\eta), & (\xi,\eta)\in Q_{\xi_0}, \\ v(\xi_0,\eta)=0, & 0<\eta<1, \\ v(\xi,0)=v(\xi,1)=0, & \xi>\xi_0, \end{cases} \tag{6}$$

where

$$f(\xi,\eta)=\psi^2 g(\xi,\eta), \tag{7}$$

and L is the second differential operator with C^{∞}-bounded coefficients on Q_{ξ_0} given by

$$\begin{aligned} & [Lv](\xi,\eta) \\ = \; & \eta^2(\psi')^2\partial^2_{\eta}v(\xi,\eta)+2\eta\psi'\partial^2_{\eta\xi}(\xi,\eta) \\ & +\psi'\partial_{\xi}v(\xi,\eta)-\eta\left(2(\psi')^2-(\psi\psi'')^2\right)\partial_{\eta}v. \end{aligned} \tag{8}$$

Remark 3 *Note here that*

$$\forall\eta\in[0,1]: \quad \lim_{\xi\to+\infty}g(\xi,\eta)=\lim_{\xi\to+\infty}h(\theta(\xi),\eta\psi(\theta(\xi)))=h(0,0). \tag{9}$$

and

$$\forall\eta\in[0,1]: \lim_{\xi\to+\infty}f(\xi,\eta)=\lim_{\xi\to+\infty}\psi^2(\theta(\xi))\,h(\theta(\xi),\eta\psi(\theta(\xi)))=0.$$

In the sequel, we will focus ourselves on the study of the concrete problem :

$$\begin{cases} \Delta_{(\xi,\eta)}v(\xi,\eta)=f(\xi,\eta), & (\xi,\eta)\in Q_{\xi_0}, \\ v(\xi_0,\eta)=0, & 0<\eta<1, \\ v(\xi,0)=v(\xi,1)=0, & \xi>\xi_0. \end{cases} \tag{10}$$

Using the same argument as in [4] and [8], one has

Proposition 4 *Let* $\sigma\in\left]0,\frac{1}{2}\right[$. *Then*

$$h\in h^{2\sigma}\left(\overline{\Omega}_{x_0}\right)\Rightarrow f\in h_b^{2\sigma}\left(\overline{Q_{\xi_0}}\right).$$

3.1. The abstract formulation of the problem (10)

Set $E = C([0,1])$ endowed with its usual norm. Define the vector-valued following functions :

$$\begin{aligned} v &: [\xi_0, +\infty[\to E;\ \xi \longrightarrow v(\xi); \quad v(\xi)(\eta) = v(\xi,\eta), \\ f &: [\xi_0, +\infty[\to E;\ \xi \longrightarrow f(\xi); \quad f(\xi)(\eta) = f(\xi,\eta). \end{aligned}$$

Consider the operator A defined by

$$\begin{cases} D(A) & = \{ w \in C^2([0,1]) : w(0) = w(1) = 0 \}, \\ (Aw)(\eta) & = D^2_\eta w(\eta). \end{cases} \tag{11}$$

Then, the concrete problem

$$\begin{cases} \Delta_{(\xi,\eta)} v(\xi,\eta) = f(\xi,\eta), & (\xi,\eta) \in Q_{\xi_0}, \\ v(\xi_0,\eta) = 0, & 0 < \eta < 1, \\ v(\xi,0) = v(\xi,1) = 0, & \xi > \xi_0, \end{cases} \tag{12}$$

is written in the following operational form

$$\begin{cases} v''(\xi) + Av(\xi) = f(\xi) & \xi > \xi_0, \\ v(\xi_0) = 0. \end{cases} \tag{13}$$

In order to obtain more optimal results for the problem (13), it will be more convenient to study the problem

$$\begin{cases} v''(\xi) + Av(\xi) = f(\xi) & \xi > \xi_0, \\ v(\xi_0) = \varphi, \end{cases} \tag{14}$$

where

$$f \in h_b^{2\sigma}\left(\overline{Q_{\xi_0}}\right), \tag{15}$$

and $\varphi \in E$.

It is well known that A it is a closed non densely defined operator satisfying the Krein-ellipticity property, that is : $\mathbb{R}^+ \subset \rho(A)$ and

$$\exists\ C > 0 : \forall \lambda \geqslant 0 \quad \left\| (A - \lambda I)^{-1} \right\|_{L(E)} \leqslant \frac{C}{1 + |\lambda|}, \tag{16}$$

(here $\rho(A)$ is the resolvent set of A).

Assumption (16) implies that operator $B = -(-A)^{1/2}$ is well defined and it is the infinitesimal generator of the generalized analytic semigroup $\left(e^{\xi B}\right)_{\xi > 0}$. More precisely, there exists a sector

$$\Pi_{\delta, r_0} = \{ z \in \mathbb{C}^* : |\arg z| \leqslant \delta + \pi/2 \} \cup \overline{B(0, r_0)},$$

(with some positive δ, r_0) and $C > 0$ such that $\rho(B) \supset \Pi_{\delta, r_0}$ and

$$\exists C > 0 : \forall z \in \Pi_{\delta, r_0}, \quad \left\| (B - zI)^{-1} \right\| \leqslant \frac{C}{1 + |z|}.$$

Remark 5 *Let us state some properties of the analytic semigroup* $\left(e^{\xi B}\right)_{\xi>0}$:

1. $\exists\omega > 0 : \forall k \in \mathbb{N}$, $\exists m_k \geq 1$ *such that*
$$\left\|\xi^k B^k e^{B\xi}\right\|_{L(E)} \leq m_k e^{-\omega\xi}, \tag{17}$$
2. $\lim\limits_{\xi\to 0} e^{\xi B}\varphi = \varphi$ *if and only if* $\varphi \in \overline{D(B)}$.

Note that in our case, one has

$$\overline{D(A)} = \{\psi \in C([0,1]) : \psi(0) = \psi(1) = 0\} = \overline{D(B)},$$

Thanks to Assumption (16), we introduce the real Banach interpolation spaces between $D(A)$ and E :

$$D_A(2\sigma) = \left\{\zeta \in E : \lim_{r\to+\infty} \left\|r^{2\sigma} A (A - rI)^{-1} \zeta\right\|_E = 0\right\}, \sigma \in \left]0, 1/2\right[.$$

More details about these Banach spaces are given in [11] and [14].
As in [2] and [13], we are studying our equation (14) under the hypothesis

$$f \in L^\infty(]\xi_0, +\infty[\,; h^{2\sigma}([0,1])) \cap h_b^{2\sigma}([\xi_0, +\infty[\,; C([0,1])). \tag{18}$$

4. Optimal results for Problem (14)

Consider the natural change of function : for $\xi \in [0, +\infty[$, set

$$V(\xi) = v(\xi + \xi_0), \qquad F(\xi) = f(\xi + \xi_0),$$

where v is the eventual solution of (14) and f satisfying (18). Therefore it is clear that

$$F \in L^\infty(]0, +\infty[\,; h^{2\sigma}([0,1])) \cap h_b^{2\sigma}([0, +\infty[\,; C([0,1])); \tag{19}$$

now the complete analysis of (14) on $[\xi_0, +\infty[$ is equivalent to the one done for the following

$$\begin{cases} V''(\xi) + AV(\xi) = F(\xi) & \xi > 0, \\ V(0) = \varphi, \end{cases} \tag{20}$$

on $[0, +\infty[$.
Let us focus ourselves on the study of the problem (20). The solution of (20) is given formally by

$$\begin{aligned} V(\xi) &= e^{B\xi}\varphi + \frac{1}{2}\int_0^{+\infty} e^{B(\xi+s)} B^{-1} F(s) ds \\ &\quad -\frac{1}{2}\int_0^{\xi} e^{B(\xi-s)} B^{-1} F(s) ds \\ &\quad -\frac{1}{2}\int_{\xi}^{+\infty} e^{B(s-\xi)} B^{-1} F(s) ds. \end{aligned} \tag{21}$$

Note that the absolute convergence of the second and fourth integral is obtained due to the estimate (17). In fact, for instance, one has

$$\begin{aligned}\left\| \int_{\xi}^{+\infty} e^{B(s-\xi)} B^{-1} F(s) ds \right\| &\leqslant C \left(\int_{\xi}^{+\infty} e^{-\omega(s-\xi)} ds \right) \max_{t\in[0,+\infty[} \|F(t)\|_E \\ &\leqslant C \max_{t\in[0,+\infty[} \|F(t)\|_E .\end{aligned}$$

We can state some regularity properties of V :

4.1. Optimal results using $F \in h_b^{2\sigma}\left([0,+\infty[;E\right)$

In this subsection we will use the fact that

$F \in h_b^{2\sigma}\left([0,+\infty[;E\right);$

recall also that F verifies

$$\lim_{\xi\to+\infty} F(\xi) = 0. \tag{22}$$

Assume that $\varphi \in D\left(A\right) = D(B^2)$. Write

$$\begin{aligned} V(\xi) &= e^{B\xi}\varphi + \frac{1}{2}\int_0^{+\infty} e^{B(\xi+s)} B^{-1} F(s) ds \\ &\quad -\frac{1}{2}\int_0^{\xi} e^{B(\xi-s)} B^{-1} F(s) ds \\ &\quad -\frac{1}{2}\int_{\xi}^{+\infty} e^{B(s-\xi)} B^{-1} F(s) ds \\ &= V_1(\xi) + V_2(\xi) + V_3(\xi) + V_4(\xi). \end{aligned} \tag{23}$$

Then

$\left\|B^2 V_1\left(\xi\right)\right\|_E = \left\|e^{B\xi} B^2\varphi\right\|_E \leq C\left\|\varphi\right\|_{D(B^2)}.$

Concerning $V_2\left(\xi\right)$, one has

$$V_2\left(\xi\right) = \frac{B^{-1}}{2}\int_0^{+\infty} e^{B(\xi+s)} F(s) ds \in D\left(B\right),$$

and

$$\begin{aligned} BV_2\left(\xi\right) &= \frac{e^{B\xi}}{2}\int_0^{+\infty} e^{Bs} F(s) ds \\ &= \frac{e^{B\xi}}{2}\int_0^{+\infty} e^{Bs}(F(s)-F(0)) ds + \frac{1}{2}B^{-1}e^{B\xi}F(0), \end{aligned}$$

from which it follows that

$$B^2V_2(\xi) = \frac{e^{B\xi}}{2}\int_0^{+\infty} Be^{Bs}(F(s)-F(0))ds + \frac{1}{2}e^{B\xi}F(0),$$

and clearly

$$\begin{aligned}\left\|B^2V_2(\xi)\right\|_E &\leqslant C\left(\int_0^{+\infty} e^{-\omega s}s^{2\sigma-1}ds\right)\|F\|_{h_b^{2\sigma}([0,+\infty[;E)} + C'\|F(0)\|_E \\ &\leqslant C\frac{\Gamma(2\sigma)}{\omega^{2\sigma}}\|F\|_{h_b^{2\sigma}([0,+\infty[;E)} + C'\|F(0)\|_E \\ &\leqslant C\|F\|_{h_b^{2\sigma}([0,+\infty[;E)},\end{aligned}$$

where Γ is the usual Euler function defined by

$$\Gamma(z) = \int_0^{+\infty} e^{-w}.w^{z-1}dw, \qquad \operatorname{Re} z > 0.$$

For $V_3(\xi)$, by writing

$$\begin{aligned}BV_3(\xi) &= -\frac{1}{2}\int_0^{\xi} e^{B(\xi-s)}F(s)ds \\ &= -\frac{1}{2}\int_0^{\xi} e^{B(\xi-s)}(F(s)-F(\xi))ds - \frac{B^{-1}}{2}e^{B\xi}F(\xi) + \frac{B^{-1}}{2}F(\xi);\end{aligned}$$

we have

$$B^2V_3(\xi) = -\frac{1}{2}\int_0^{\xi} Be^{B(\xi-s)}(F(s)-F(\xi))ds - \frac{1}{2}e^{B\xi}F(\xi) + \frac{1}{2}F(\xi);$$

thus

$$\left\|B^2V_3(\xi)\right\|_E \leqslant C\|F\|_{h_b^{2\sigma}([0,+\infty[;E)} + C'\|F(\xi)\|_E \leqslant C''\|F\|_{h_b^{2\sigma}([0,+\infty[;E)}.$$

Finally

$$\begin{aligned}&BV_4(\xi) \\ = &-\frac{1}{2}\int_{\xi}^{+\infty} e^{B(s-\xi)}F(s)ds \\ = &-\frac{1}{2}\int_{\xi}^{+\infty} e^{B(s-\xi)}(F(s)-F(\xi))ds + \frac{B^{-1}}{2}F(\xi),\end{aligned}$$

and

$$B^2 V_4(\xi) = -\frac{1}{2}\int_{\xi}^{+\infty} Be^{B(s-\xi)}\left(F(s) - F(\xi)\right)ds + \frac{1}{2}F(\xi),$$

which gives the estimate

$$\left\|B^2 V_4(\xi)\right\|_E \leqslant C\left\|F\right\|_{h_b^{2\sigma}([0,+\infty[;E)} + C'\left\|F(\xi)\right\|_E \leqslant C''\left\|F\right\|_{h_b^{2\sigma}([0,+\infty[;E)}.$$

Summarizing, we obtain the following decomposition

$$\begin{aligned} & B^2V(\xi) \\ = \; & e^{B\xi}\left[B^2\varphi + F(0)\right] + F(\xi) \\ & + \frac{e^{B\xi}}{2}\left(\int_0^{+\infty} Be^{Bs}(F(s) - F(0))ds\right) \\ & - \frac{1}{2}\int_0^{\xi} Be^{B(\xi-s)}(F(s) - F(\xi))ds \\ & - \frac{1}{2}\int_{\xi}^{+\infty} Be^{B(s-\xi)}\left(F(s) - F(\xi)\right)ds. \end{aligned} \tag{24}$$

Proposition 6 *Let V given in (23). Then* $\lim\limits_{\xi\to+\infty} V(\xi) = 0.$

Proof. We know that there exists $\omega > 0$ and $m_0 > 0$ such that for any $\xi > 0$

$$\left\|V_1(\xi)\right\|_E \leq m_0 e^{-\omega\xi}\left\|\varphi\right\|_E,$$

then

$$\lim_{\xi\to+\infty}\left\|V_1(\xi)\right\|_E = 0.$$

the same is true for $V_2(\xi)$.
For $V_3(\xi)$, one write

$$\begin{aligned} & V_2(\xi) \\ = \; & -\frac{1}{2}\left(\int_0^{\xi/2} e^{B(\xi-s)}B^{-1}F(s)ds + \int_{\xi/2}^{\xi} e^{B(\xi-s)}B^{-1}F(s)ds\right) \\ = \; & -\frac{1}{2}\left(V_{21}(\xi) + V_{22}(\xi)\right). \end{aligned}$$

One has

$$\begin{aligned}\|V_{21}(\xi)\|_E &\leq C\left(\int\limits_0^{\xi/2} e^{-\omega(\xi-s)}ds\right)\|F\|_{h_b^{2\sigma}([0,+\infty[;E)}\\ &\leq \frac{C}{\omega}\left(e^{-\omega\xi/2}-e^{-\omega\xi}\right)\|F\|_{h_b^{2\sigma}([0,+\infty[;E)},\end{aligned}$$

consequently,

$$\lim_{\xi\to+\infty} V_{21}(\xi)=0.$$

Since $\lim\limits_{\xi\to+\infty} F(\xi)=0$, then

$$\lim_{\xi\to+\infty}\sup_{\frac{\xi}{2}\leq s\leq\xi}\|F(s)\|_E=0.$$

So,

$$\begin{aligned}\|V_{22}(\xi)\|_E &\leq C\sup_{\frac{\xi}{2}\leq s\leq\xi}\|F(s)\|_E\int\limits_{\xi/2}^{\xi}\left\|e^{B(\xi-s)}\right\|_{L(E)}ds\\ &\leq C\left(1-e^{-\omega\xi/2}\right)\left(\sup_{\frac{\xi}{2}\leq s\leq\xi}\|f(s)\|_E\right).\end{aligned}$$

Therefore,

$$\lim_{\xi\to+\infty}\|V_{22}(\xi)\|_E=0.$$

In the same way we obtain

$$\lim_{\xi\to+\infty} V_3(\xi)=0,\qquad \lim_{\xi\to+\infty} V_4(\xi)=0.$$

We have the following result summarizing the complete analysis of V for $F\in h_b^{2\sigma}([0,+\infty[;E)$.

Proposition 7 *Let $\varphi\in D(A)$. Then V given in (23) is the unique solution of (20) satisfying*

1. *V'', $AV(.)\in C_b([0,+\infty[;E)$ if and only if $F(0)-A\varphi=F(0)+B^2\varphi\in\overline{D(A)}$,*
2. *V'', $AV(.)\in h_b^{2\sigma}([0,+\infty[;E)$ if and only if $F(0)-A\varphi=F(0)+B^2\varphi\in D_A(\sigma)$*

Sketch of the proof. Recall that

$$\begin{aligned} & B^2V(\xi) \\ = \; & e^{B\xi}\left(B^2\varphi + F(0)\right) \\ & + \frac{e^{B\xi}}{2}\left(\int_0^{+\infty} Be^{Bs}(F(s)-F(0))ds\right) \\ & - \frac{1}{2}\int_0^{\xi} Be^{B(\xi-s)}(F(s)-F(\xi))ds \\ & - \frac{1}{2}\int_{\xi}^{+\infty} Be^{B(s-\xi)}\left(F(s)-F(\xi)\right)ds \\ & + F(\xi). \end{aligned} \tag{25}$$

The proof is essentially based on the representation (24) and all the properties proved in [14], in particular see Proposition 1.2, p. 20 and Theorem 4.5, p. 53. For instance, the term

$$e^{B\xi}\left(B^2\varphi + F(0)\right),$$

is continuous at 0 if and only if $F(0) + B^2\varphi \in \overline{D(A)}$ and its limit when $\xi \to 0^+$ is $B^2\varphi + F(0)$. The second term writes

$$\begin{aligned} & \frac{e^{B\xi}}{2}\left(\int_0^{+\infty} Be^{Bs}(F(s)-F(0))ds\right) \\ = \; & \frac{e^{B\xi}}{2}\left(\int_0^{1} Be^{Bs}(F(s)-F(0))ds + \int_1^{+\infty} Be^{Bs}(F(s)-F(0))ds\right) \\ = \; & \frac{e^{B\xi}}{2}\left[(a)+(b)\right], \end{aligned}$$

it is well known that $(a) \in D_A(\sigma, +\infty)$ when F is only in $C_b^{2\sigma}\left([0,+\infty[;E\right)$ and $(a) \in D_A(\sigma)$ in our case $F \in h_b^{2\sigma}\left([0,+\infty[;E\right)$ while (b) is very regular since it belongs to $D(B^k)$ for all $k \in \mathbb{N}^*$. In the same way we analyze the other integrals in $B^2V(\xi)$.
Remark also that each term in (25) tends to 0 when $\xi \to +\infty$; the proof is the same as for V.

Going back to our operational problem (14), one obtains obviously, for $\xi \geqslant \xi_0$

$$\begin{aligned} v(\xi) &= V(\xi-\xi_0) = e^{B(\xi-\xi_0)}\varphi + \frac{1}{2}\int_0^{+\infty} e^{B(\xi-\xi_0+s)}B^{-1}F(s)ds \\ &-\frac{1}{2}\int_0^{\xi-\xi_0} e^{B(\xi-\xi_0-s)}B^{-1}F(s)ds \\ &-\frac{1}{2}\int_{\xi-\xi_0}^{+\infty} e^{B(s-\xi+\xi_0)}B^{-1}F(s)ds, \end{aligned} \tag{26}$$

and the following result

Proposition 8 *Let $\varphi \in C^2([0,1])$ such that $\varphi(0) = \varphi(1) = 0$. Then v given in (26) is the unique solution of (14) satisfying*

1. $\dfrac{\partial^2 v}{\partial \xi^2}, \dfrac{\partial^2 v}{\partial \eta^2} \in C_b([\xi_0, +\infty[\,; C([0,1]))$ *if and only if*
$$\begin{cases} \eta \longmapsto f(\xi_0,\eta) - \varphi''(\eta) \in C([0,1]) \text{ and} \\ f(\xi_0,0) - \varphi''(0) = f(\xi_0,1) - \varphi''(1) = 0. \end{cases}$$
2. $\dfrac{\partial^2 v}{\partial \xi^2}, \dfrac{\partial^2 v}{\partial \eta^2} \in h_b^{2\sigma}([\xi_0, +\infty[; C([0,1]))$ *if and only if*
$$\begin{cases} \eta \longmapsto f(\xi_0,\eta) - \varphi''(\eta) \in h^{2\sigma}([-1,1]) \text{ and} \\ f(\xi_0,0) - \varphi''(0) = f(\xi_0,1) - \varphi''(1) = 0. \end{cases}$$

Remark 9 *Let $\varphi = 0$. We can also give the following representation of the solution by using the Grisvard method, see [5]*

$$\begin{aligned} V(\xi) &= -\frac{1}{2i\pi}\int_{\gamma_1}\int_0^{\xi} \frac{e^{-\sqrt{-z}\xi}\sinh\sqrt{-z}s}{\sqrt{-z}}(A-zI)^{-1}F(s)dsdz \\ &-\frac{1}{2i\pi}\int_{\gamma_1}\int_{\xi}^{+\infty} \frac{e^{-\sqrt{-z}s}\sinh\sqrt{-z}\xi}{\sqrt{-z}}(A-zI)^{-1}F(s)dsdz, \end{aligned}$$

where γ_1 is the boundary of

$S(\omega_0,\epsilon_0) = \{z \in \mathbb{C}^* : |\arg z| \le \omega_0\} \cup B(0,\epsilon_0)$.

Thus

$$\begin{aligned} V(\xi,\eta) &= V(\xi)(\eta) \\ &= -\frac{1}{2i\pi}\int_{\gamma_1}\int_0^{\xi} \frac{e^{-\sqrt{-z}\xi}\sinh\sqrt{-z}s}{\sqrt{-z}}\left((A-zI)^{-1}F(s)\right)(\eta)dsdz \\ &\quad -\frac{1}{2i\pi}\int_{\gamma_1}\int_{\xi}^{+\infty} \frac{e^{-\sqrt{-z}s}\sinh\sqrt{-z}\xi}{\sqrt{-z}}\left((A-zI)^{-1}F(s)\right)(\eta)dsdz; \end{aligned}$$

or equivalently

$$\begin{aligned}
v(\xi,\eta) &= v(\xi)(\eta)=V(\xi-\xi_0)(\eta)\\
&= -\frac{1}{2i\pi}\int_{\gamma_1}\int\limits_0^{\xi-\xi_0}\frac{e^{-\sqrt{-z}(\xi-\xi_0)}\sinh\sqrt{-z}s}{\sqrt{-z}}\left((A-zI)^{-1}F(s)\right)(\eta)dsdz\\
&\quad -\frac{1}{2i\pi}\int_{\gamma_1}\int\limits_{\xi-\xi_0}^{+\infty}\frac{e^{-\sqrt{-z}s}\sinh\sqrt{-z}(\xi-\xi_0)}{\sqrt{-z}}\left((A-zI)^{-1}F(s)\right)(\eta)dsdz;
\end{aligned}$$

but in our case, one has

$$\begin{aligned}
&\left((A-zI)^{-1}F(s)\right)(\eta)\\
&= -\int_0^{\eta}\frac{\sinh\sqrt{z}\,(1-\eta)\sinh\tau\sqrt{z}}{\sqrt{z}\sinh\sqrt{z}}F(s)(\tau)d\tau\\
&\quad -\int_{\eta}^{1}\frac{\sinh\sqrt{z}\,(1-\tau)\sinh\eta\sqrt{z}}{\sqrt{z}\sinh\sqrt{z}}F(s)(\tau)d\tau\\
&= \int_0^1 K_z(\eta,\tau)F(s)(\tau)d\tau\\
&= \int_0^1 K_z(\eta,\tau)f(s+\xi_0,\tau)d\tau,
\end{aligned}$$

with some natural modification for $(A-zI)^{-1}$ near zero (which is deduced from the explicit calculus of A^{-1}).
We then obtain the formula

$$\begin{aligned}
&v(\xi,\eta)\\
&= -\frac{1}{2i\pi}\int_{\gamma_1}\int\limits_0^{\xi-\xi_0}\frac{e^{-\sqrt{-z}(\xi-\xi_0)}\sinh\sqrt{-z}s}{\sqrt{-z}}\left[\int_0^1 K_z(\eta,\tau)f(s+\xi_0,\tau)d\tau\right]dsdz\\
&\quad -\frac{1}{2i\pi}\int_{\gamma_1}\int\limits_{\xi-\xi_0}^{+\infty}\frac{e^{-\sqrt{-z}s}\sinh\sqrt{-z}(\xi-\xi_0)}{\sqrt{-z}}\left[\int_0^1 K_z(\eta,\tau)f(s+\xi_0,\tau)d\tau\right]dsdz.
\end{aligned}$$

4.2. Optimal results using $F\in L^{\infty}(]0,+\infty[\,;h^{2\sigma}([0,1]))$

In this case, we decompose our problem

$$\begin{cases}
\Delta_{(\xi,\eta)}v(\xi,\eta)=f(\xi,\eta), & (\xi,\eta)\in Q_{\xi_0},\\
v(\xi_0,\eta)=\varphi(\eta), & 0<\eta<1,\\
v(\xi,0)=v(\xi,1)=0, & \xi>\xi_0,
\end{cases}\tag{27}$$

in the two following problems

$$\begin{cases}
\Delta_{(\xi,\eta)}v_1(\xi,\eta)=0=f_1(\xi,\eta), & (\xi,\eta)\in Q_{\xi_0},\\
v_1(\xi_0,\eta)=\varphi(\eta), & 0<\eta<1,\\
v_1(\xi,0)=v_1(\xi,1)=0, & \xi>\xi_0,
\end{cases}\tag{28}$$

and

$$\begin{cases} \Delta_{(\xi,\eta)} v_2(\xi,\eta) = f(\xi,\eta), & (\xi,\eta) \in Q_{\xi_0}, \\ v_2(\xi_0,\eta) = 0, & 0 < \eta < 1, \\ v_2(\xi,0) = v_2(\xi,1) = 0, & \xi > \xi_0. \end{cases} \tag{29}$$

As above, clearly (28) leads to

$$\begin{cases} V_1''(\xi) + AV_1(\xi) = F_1(\xi) = 0 \quad \xi > 0, \\ V_1(0) = \varphi, \end{cases} \tag{30}$$

and here

$$F_1 = 0 \in C_b([0,+\infty[, h_0^{2\sigma}([0,1])) = C([0,+\infty[; D_A(\sigma))$$

since, one has exactly

$$D_A(\sigma) = \left\{\phi \in h^{2\sigma}([0,1]) : \phi(0) = \phi(1) = 0\right\} = h_0^{2\sigma}([0,1]).$$

Therefore we can use for the abstract equation (30), the same operational techniques as above and due to [14], see Theorem 5.5, p. 60. We then obtain the following result.

Proposition 10 *Let $\varphi \in D(A)$. Then there exists a unique solution V_1 to (30) satisfying*

1. V_1'', $AV_1(.) \in C_b([0,+\infty[\,;E)$
2. *if* $A\varphi \in D_A(\sigma)$ *then* V_1'', $AV_1(.) \in h_b^{2\sigma}([0,+\infty[; E) \cap L^\infty(]0,+\infty[\,; h^{2\sigma}([0,1]))$.

The same results hold true for v_1 which gives

Proposition 11 *Let $\varphi \in C^2([0,1])$ such that $\varphi(0) = \varphi(1) = 0$. Then there exists a unique solution v_1 to (28) satisfying*

1. $\dfrac{\partial^2 v_1}{\partial \xi^2}, \dfrac{\partial^2 v_1}{\partial \eta^2} \in C_b([\xi_0,+\infty[\,;E)$
2. *if*

$$\eta \longmapsto \varphi''(\eta) \in h^{2\sigma}([0,1]) \text{ and } \varphi''(0) = \varphi''(1) = 0,$$

$$\text{then } \frac{\partial^2 v_1}{\partial \xi^2}, \frac{\partial^2 v_1}{\partial \eta^2} \in h_b^{2\sigma}(]\xi_0,+\infty]\,;E) \cap L^\infty(]\xi_0,+\infty[\,;h^{2\sigma}([0,1])).$$

It remains to analyze (29) with

$$f \in L^\infty(]\xi_0,+\infty[\,;h^{2\sigma}([0,1]))$$

without boundary condition on $f(\xi,.)$ at 0 and 1. One must invert the abstract writing of (29) in order to use the regularity with respect to η. We write (29) in the following form

$$\begin{cases} V_2''(\eta) + A_2V_2(\eta) = F(\eta) \quad 0 < \eta < 1, \\ V_2(0) = V_2(1) = 0, \end{cases} \tag{31}$$

in the space $E_2 = L^\infty([\xi_0, +\infty[)$ where for all $\eta \in [0,1]$ and for *a.e.* $\xi \in]\xi_0, +\infty]$

$$F(\eta) : \eta \longmapsto F(\eta)(\xi) = f(\xi, \eta), \quad V_2(\eta) : \xi \longmapsto V_2(\eta)(\xi) = v_2(\xi, \eta),$$

and A_2 is the closed linear operator

$$\begin{cases} D(A_2) = \{\psi \in W^{2,\infty}(]\xi_0, +\infty]) : \psi(\xi_0) = 0\} \\ (A_2\psi)(\xi) = \psi''(\xi). \end{cases}$$

Note that here also $D(A_2)$ is not dense in $E = L^\infty(]\xi_0, +\infty])$.

As for A, operator A_2 satisfies the Krein-ellipticity property, that is : $\mathbb{R}^+ \subset \rho(A_2)$ and $\exists\, C > 0 : \forall \lambda \geqslant 0$

$$\left\|(A_2 - \lambda I)^{-1}\right\|_{L(E)} \leqslant \frac{C}{1 + |\lambda|},$$

which implies that operator $B_2 = -(-A_2)^{1/2}$ is well defined and is the infinitesimal generator of the generalized analytic semigroup $\left(e^{\xi B_2}\right)_{\xi > 0}$.

Therefore the techniques used in the above subsection apply and one obtains

Proposition 12 *There exists a unique solution V_2 of (31) satisfying*

1. V_2'', $A_2 V_2(.) \in C([0,1]; E_2)$ *if and only if* $F(0), F(1) \in \overline{D(A_2)}$,
2. V_2'', $AV_2(.) \in h^{2\sigma}([0,1]; E_2)$ *if and only if* $F(0), F(1) \in D_{A_2}(\sigma)$.

This Proposition has the equivalent for v_2 by inverting the variables (η, ξ)

Proposition 13 *There exists a unique solution $v_2(\xi, \eta)$ of (27) satisfying*

1. $\dfrac{\partial^2 v_2}{\partial \eta^2}, \dfrac{\partial^2 v_2}{\partial \xi^2} \in L^\infty(]\xi_0, +\infty]\,; C([0,1]))$ *if and only if*
$$\begin{cases} \xi \mapsto f(\xi, 0), \ f(\xi, 1) \in C_b([\xi_0, +\infty[) \\ f(\xi_0, 0) = f(\xi_0, 1) = 0, \end{cases}$$
2. $\dfrac{\partial^2 v_2}{\partial \eta^2}, \dfrac{\partial^2 v_2}{\partial \xi^2} \in L^\infty(]\xi_0, +\infty]\,; h^{2\sigma}([0,1]))$ *if and only if*
$$\xi \mapsto f(\xi, 0), f(\xi, 1) \in D_{A_2}(\sigma).$$

Therefore we summarize the results in the case of $F \in L^\infty(]\xi_0, +\infty]\,; h^{2\sigma}([0,1]))$ by writing for $v = v_1 + v_2$

Proposition 14 *Let $\varphi \in C^2([0,1])$ such that $\varphi(0) = \varphi(1) = 0$. Then there exists a unique solution v such that if*

$$\begin{cases} \eta \longmapsto \varphi''(\eta) \in h^{2\sigma}([0,1]) \ \text{and}\ \varphi''(0) = \varphi''(1) = 0 \ \text{and} \\ \xi \mapsto f(\xi, 0), \ f(\xi, 1) \in D_{A_2}(\sigma), \end{cases}$$

then $\dfrac{\partial^2 v}{\partial \xi^2}, \dfrac{\partial^2 v}{\partial \eta^2} \in L^\infty(]\xi_0, +\infty]\,; h^{2\sigma}([0,1]))$.

4.3. Complete results using $f \in h^{2\sigma}(\overline{Q_{\xi_0}})$

Now we are in position to summarize all the results concerning our Problem (12) with

$$f \in h_b^{2\sigma}\left([\xi_0, +\infty[; E\right) \cap L^{\infty}\left(]\xi_0, +\infty] ; h^{2\sigma}([0,1])\right) = h_b^{2\sigma}(\overline{Q_{\xi_0}}).$$

From the results in the above subsections, one obtains

Proposition 15 *Let $\varphi \in C^2([0,1])$ such that $\varphi(0) = \varphi(1) = 0$. Then there exists a unique solution v of (10) such that*

1. *if*

$$\begin{cases} \eta \longmapsto \varphi''(\eta) \in h^{2\sigma}([0,1]) \ \text{ and } \varphi''(0) = \varphi''(1) = 0 \text{ and} \\ \xi \mapsto f(\xi, 0), \ f(\xi, 1) \in D_{A_2}(\sigma), \end{cases}$$

then $\dfrac{\partial^2 v}{\partial \xi^2}, \dfrac{\partial^2 v}{\partial \eta^2} \in L^{\infty}(]\xi_0, +\infty[\, ; h^{2\sigma}([0,1]))$.

2. $\dfrac{\partial^2 v}{\partial \xi^2}, \dfrac{\partial^2 v}{\partial \eta^2} \in h_b^{2\sigma}([\xi_0, +\infty[; C([0,1]))$ *if and only if*

$$\begin{cases} \eta \longmapsto f(\xi_0, \eta) - \varphi''(\eta) \in h^{2\sigma}([0,1]) \ \text{ and} \\ f(\xi_0, 0) - \varphi''(0) = f(\xi_0, 1) - \varphi''(1) = 0. \end{cases}$$

Let us focus ourselves on the case when

$$\varphi = 0,$$

recall that the conditions

$$\xi \mapsto f(\xi, 0), \ f(\xi, 1) \in D_{A_2}(\sigma), \tag{32}$$

mean that

$$f(\xi_0, 0) = f(\xi_0, 1) = 0,$$

then, one has

Proposition 16 *Let $f \in h^{2\sigma}(\overline{Q_{\xi_0}})$, ($\sigma \in]0, 1/2[$) such that*

$$\xi \mapsto f(\xi_0, 0) = \ f(\xi_0, 1) = 0. \tag{33}$$

Then there exists a unique solution v of (12) such that

$$\frac{\partial^2 v}{\partial \xi^2}, \frac{\partial^2 v}{\partial \eta^2} \in L^{\infty}(]\xi_0, +\infty[\, ; h^{2\sigma}([0,1])) \cap h_b^{2\sigma}([\xi_0, +\infty[; C([0,1])) = h_b^{2\sigma}(\overline{Q_{\xi_0}}).$$

Set

$$X_d = \left\{ v \in h^{2+2\sigma}(\overline{Q_{\xi_0}}) : v = 0 \text{ on } \partial\overline{Q_{\xi_0}} \right\} \tag{34}$$

and

$$X_a = \left\{ f \in h_b^{2\sigma}(\overline{Q_{\xi_0}}) : f(\xi_0, 0) = f(\xi_0, 01) = 0 \right\}. \tag{35}$$

Therefore, we deduce that the Laplace operator

$$\Delta : \; X_d \;\to\; X_a,$$

is an isomorphism.

At this level, we recall that we look to study the regularity of the solution of Problem (6) at the vicinity of D_∞ (given by 4). For this reason, we introduce the two following operators

–

$$\begin{array}{llll} T : & h_b^{2\sigma}(\overline{Q_{\xi_0}}) & \to & h_b^{2\sigma}(\overline{Q_{\xi_0}}) \\ & f & \mapsto & k(\xi) f, \end{array} \tag{36}$$

where $k : \mathbb{R}^+ \to \mathbb{R}$ is the the truncation function defind by

$$\begin{cases} k(\xi) = 0 & 0 \leq \xi \leq 2\xi_0 := \xi_1, \\ k(\xi) = \xi - \xi_1 & \xi_1 \leq \xi \leq 2\xi_1 := \xi_{2,} \\ k(\xi) = 1 & \xi \geq \xi_2. \end{cases}$$

–

$$\begin{array}{llll} \overline{L} : & h_b^{2+2\sigma}(\overline{Q_{\xi_0}}) & \to & h_b^{2\sigma}(\overline{Q_{\xi_0}}) \\ & f & \mapsto & (T \circ L) f \end{array} \tag{37}$$

where L is given by (8).

Lemma 17 *Let $\sigma \in]0, 1/2[$ One has*

1. *The linear operator T is continuous with*

$$\|Tf\|_{h^{2\sigma}(\overline{Q_{\xi_0}})} \leq 2 \|f\|_{h^{2\sigma}(\overline{Q_{\xi_0}})}.$$

2. *The linear operator $\overline{L}$ is continuous with*

$$\left\|\overline{L}f\right\|_{h^{2+2\sigma}(\overline{Q_{\xi_0}})} \leq 2 \|L\|_{h^{2+2\sigma}(\overline{Q_{\xi_0}})}.$$

Using the same argument as in [7] and Keeping in mind the results of the previous section. We deduce that there exist $\xi^* \geq \xi_2$ large enough such that

$$\Delta + \overline{L} : \; X_d \;\to\; X_a$$

is an isomorphism. This justifies our main result concerning our complete transformed problem (6)

Proposition 18 *Let* $f \in h_b^{2\sigma}(\overline{Q_{\xi_0}})$, $(\sigma \in]0,1/2[)$ *such that*

$\xi \mapsto f(\xi_0, 0) = \ f(\xi_0, 1) = 0.$

Then there exists $\xi^* \geq \xi_2$ *such that (6) admits a unique strict solution* v *satisfying*

$$\frac{\partial^2 v}{\partial \xi^2}, \frac{\partial^2 v}{\partial \eta^2} \in h_b^{2\sigma}(\overline{Q_{\xi^*}}).$$

Remark 19 *In the sequel, for simplicity, we assume that* $\xi^* = \xi_2$.

4.4. Go back to the original problem

Let $\overline{u}$ the unique variationnal solution of Problem (1), see [7]. Set

$$u = \Theta(x)\,\overline{u}$$

where $\Theta(x) \in C^\infty([0, x_0])$ such that

$$\begin{cases} 0 \leq \Theta \leq 1 \\ \Theta(x) = 0 \quad x > x_1 := \Pi^{-1}(\xi_1) \\ \Theta(x) = 1 \quad x \leq x_2 := \Pi^{-1}(\xi_2) \end{cases}$$

Remark 20 *It is easy to see that :*

1. $\begin{cases} u = 0 & x > x_1 \\ u = \overline{u} & 0 < x < x_2 \end{cases}$
2. $\Delta u \in h^{2\sigma}(\Omega_{x_0})$.

Taking into account, the resuts of the prevoius section, we conclude that for $x \leq x_2$, one has

$$\overline{u} = \Pi^{-1}(v)$$

where v is the unique solution of (6).

Now, we give the following result describing the effect of the inverse change of variables.

Lemma 21 *Let* $\sigma \in]0, 1/2[$. *Then*

$$g \in h_b^{2\sigma}\left(\overline{Q_{\xi_0}}\right) \Rightarrow (\psi(x))^{2\sigma} h \in h^{2\sigma}\left(\overline{\Omega_{x_0}}\right).$$

Recall that for all $(x, y) \in \overline{\Omega}$

$$h(x, y) = g\left(\theta^{-1}(x), \frac{y}{\psi(x)}\right).$$

Given a small $\delta > 0$. Let $(x_2, y_2), (x_1, y_1) \in \Omega_{x_0}$ such that

$$(x_2, y_2) \neq (x_1, y_1),$$

and $\|(x_2-x_1,y_2-y_1)\|\leq\delta$. Assume, for instance that $x_1\leq x_2$, $y_1\leq y_2$. First, it is easy to see that

$$\sup_{(x,y)\in\overline{\Omega_{x_0}}}\left|(\psi(x))^{2\sigma}h(x,y)\right|<\infty.$$

One has

$$\begin{aligned}
&(\psi(x_2))^{2\sigma}h(x_2,y_2)-(\psi(x_1))^{2\sigma}h(x_1,y_1)\\
=\;&\left((\psi(x_2))^{2\sigma}-(\psi(x_1))^{2\sigma}\right)h(x_2,y_2)+(\psi(x_1))^{2\sigma}(h(x_2,y_2)-h(x_1,y_1))\\
=\;&P_1+P_2.
\end{aligned}$$

Then

$$\lim_{\delta\to 0}\sup_{\|(x_2,y_2)-(x_1,y_1)\|\leq\delta}\frac{|P_1|}{\|(x_2-x_1,y_2-y_1)\|^{2\sigma}}=0.$$

For P_2, one has

$$\begin{aligned}
&\frac{|P_2|}{\|(x_2-x_1,y_2-y_1)\|^{2\sigma}}\\
=\;&(\psi(x_1))^{4\sigma}\frac{|g\circ\Pi(x_2,y_2)-g\circ\Pi(x_1,y_1)|}{\|\Pi(x_2,y_2)-\Pi(x_1,y_1)\|^{2\sigma}}\frac{\|\Pi(x_2,y_2)-\Pi(x_1,y_1)\|^{2\sigma}}{\|(x_2-x_1,y_2-y_1)\|^{2\sigma}}.
\end{aligned}$$

From $g\in h^{2\sigma}\left(\overline{Q_{\xi_0}}\right)$, we get

$$\lim_{\delta\to 0}\sup_{\|\Pi(x_2,y_2)-\Pi(x_1,y_1)\|\leq\delta}\frac{|g\circ\Pi(x_2,y_2)-g\circ\Pi(x_1,y)|}{\|\Pi(x_2,y_2)-\Pi(x_1,y_1)\|^{2\sigma}}=0.$$

It remains to estimate the second fraction ; one has

$$\begin{aligned}
&\|\Pi(x_2,y_2)-\Pi(x_1,y_1)\|^{2\sigma}\\
\leq\;&\left\|\left(\theta^{-1}(x_2)-\theta^{-1}(x_1),\frac{y_2}{\psi(x_2)}-\frac{y_1}{\psi(x_1)}\right)\right\|^{2\sigma}\\
\leq\;&\left\|\left(\theta^{-1}(x_2)-\theta^{-1}(x_1),\frac{y_2\psi(x_1)-y_1\psi(x_2)}{\psi(x_2)\psi(x_1)}\right)\right\|^{2\sigma}\\
\leq\;&\left\|\left(\theta^{-1}(x_2)-\theta^{-1}(x_1),y_2\frac{(\psi(x_1)-\psi(x_2))}{\psi(x_2)\psi(x_1)}+\psi(x_2)\frac{(y_2-y_1)}{\psi(x_2)\psi(x_1)}\right)\right\|^{2\sigma}
\end{aligned}$$

since $x_1< x_2$ Thus

$$\|\Pi(x_2,y_2)-\Pi(x_1,y_1)\|^{2\sigma}\leqslant\frac{C}{(\psi(x_1))^{2\sigma}}\|(x_2-x_1,y_2-y_1)\|^{2\sigma}$$

from which we deduce that

$$\lim_{\delta\to 0}\sup_{\|(x_2-x_1,y_2-y_1)\|\leq\delta}\frac{|P_2|}{\|(x_2-x_1,y_2-y_1)\|^{2\sigma}}=0.$$

Summing up, we get

$$\lim_{\delta\to 0}\sup_{\|(x_2-x_1,y_2-y_1)\|\leq\delta}\frac{|h(x_2,y_2)-h(x_1,y_1)|}{\|(x_2-x_1,y_2-y_1)\|^{2\sigma}}=0.$$

Now, taking into account this result and using the same techniques as in [7], we give our main result

Theorem 22 *Let $h \in h^{2\sigma}(\overline{\Omega})$, ($\sigma \in]0, 1/2[$) satisfying (3). Then, there exists $x_2 < x_0$ such that Problem (1)-(2) admits a unique strict solution u satisfying*

$$(\psi(x))^{2\sigma}\,\partial_y^2 u \text{ and } (\psi(x))^{2\sigma}\,\partial_x^2 u \in h^{2\sigma}(\overline{\Omega_{x_2}}).$$

where

$$\Omega_{x_2} := \left\{(x,y)\in\mathbb{R}^2 : 0 < x < x_2,\ 0 < y < \psi(x)\right\}. \tag{38}$$

REFERENCES

[1] K. Belahdji, La régularité L^p de la Solution du Problème de Dirichlet dans un Domaine à Points de Rebroussement, C. R. Acad. Sci. Paris, Sér. I 322 (1996), 5–8.

[2] T. Berroug, D. Hua, R. Labbas, B. K Sadallah, On a Degenerate Parabolic Problem in Hölder Spaces. Applied Mathematics and Computation, vol. 162, Issue 2 (2005), 811-833.

[3] D. L. Burkholder, A geometrical Characterisation of Banach Spaces in which Martingale Difference Sequences are Unconditional, Ann. Probab, 9 (1981), 997-1011.

[4] B. Chaouchi, R. Labbas, B. K. Sadallah, Laplace Equation on a Domain With a Cuspidal Point in Little Hölder Spaces, to appear in Mediterranean Journal of Mathematics.

[5] G Da Prato, P Grisvard, Sommes d'opérateurs linéaires et équations différentielles opérationnelles, J. Math. Pures Appl. (9) 54 (1975), no. 3, 305–387.

[6] G. Dore, A. Venni, An Operational Method to Solve a Dirichlet Problem for the Laplace Operator in a Plane Sector, Differential and integral equations, Volume 3, Number 2,(1990), 323-334.

[7] P. Grisvard, Problèmes aux Limites dans des Domaines avec Points de Rebroussement, Partial Differential Equations and Functional Analysis, Progress in Nonlinear Differential Equations Appl, 22, Birkhäuser Boston, Boston, MA (1996).

[8] P. Guidotti, 2-D Free Boundary Value Problem with Onset of a Phase and Singular Elliptic Boundary Value Problems, Journal of Evolution Equations, Vol. 2 (4) 2002, 395-424.

[9] K. Ibuki, Dirichlet problem for elliptic equations of the second order in a singular domain of R2, Journal Math. Kyoto Univ. 14, n° 1 (1974), pp. 54-71.

[10] A. Khelif, Problèmes aux limites pour le laplacien dans un domaine à points cuspides, CRAS, Paris, 287 (1978), pp. 1113-1116.

[11] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, (1995).

[12] V. Maz'ya, A. Soloviev, Boundary Integral Equations on Contours with Peaks, Operator Theory, Advances and Applications, Vol. 196, Birkhauser, 2010.

[13] M. Najmi, Régularité-Singularité dans les Espaces de Hölder pour un Problème Elliptique et l'Equation de la Chaleur dans des Domaines Non Réguliers, Thèse d'état, Université de Nice, (1992).

[14] E. Sinestrari, On the Abstract Cauchy Problem of Parabolic Type in Spaces of Continuous Functions, J. Math. Anal. Appli, 66 (1985), 16-66.